Resonance & Mutual Inductance

Resonance

You are probably familiar with Resonance in sound. In this we know that at resonance we here the greatest sound (water column) and have the maximum vibration (string). The same idea is present in electrical engineering. Resonance basically occurs when a quantity, such as voltage or current, becomes a maximum. However, a maximum with one quantity generally corresponds to a minimum with some other quantity, so that it could also correspond to the condition when minimum value of a quantity occurs.

For example, if we consider a series circuit, maximum current for a given source voltage would occur when the impedance of the circuit is a minimum. Also, if it is an a.c. circuit, the impedance ***R+jX*** has a magnitude $\sqrt{R^2 + X^2}$ which would have a minimum value when X is zero (this is possible in a practical circuit because inductive reactances and capacitive reactances have opposite signs). When X is zero, the circuit is purely resistive and the power factor of the circuit becomes unity. Thus resonance is also defined in terms of the power factor of a circuit becoming unity.

Thus there are three main methods of defining the resonance condition in an electrical circuit.

(a) When the *current* through a circuit for a given source voltage becomes a *maximum*:

This condition can also be stated in the following manner:

when the voltage across a circuit for a given source current becomes a minimum,

when the admittance of the circuit becomes a maximum, or

when the impedance of the circuit becomes a minimum.

(b) When the *voltage* across a circuit for a given source current becomes a *maximum*:

This condition can also be stated in the following manner:

when the current through a circuit for a given source voltage becomes a minimum,

when the impedance of the circuit becomes a maximum, or

when the admittance of the circuit becomes a minimum.

and

(c) When the *power factor* of the circuit becomes *Unity*:

This condition can also be stated in the following manner:

when the impedance of the circuit is purely real,

when the admittance of the circuit is purely real, or

when the voltage and the current are in phase.

The condition (a) occurs in series circuits, and is usually referred to as ***Series Resonance.***

The condition (b) occurs in parallel circuits, and is usually referred to as ***Parallel Resonance.***

While the conditions series resonance and parallel resonance are exclusive conditions, the unity power factor condition could correspond to either series resonance or parallel resonance. In complicated circuits, the latter condition could also give displaced answers from the other two conditions.

When does oscillations occur ? What determines the natural frequency of the oscillations ?

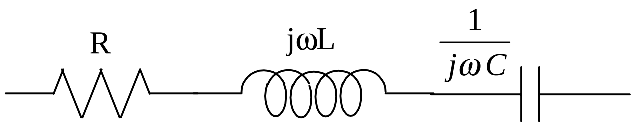
To answer this question, let us look at the simple pendulum, which you are all familiar with. Why does it oscillate ? Because there is energy in the system, and because there are two forms of energy – namely, potential energy and kinetic energy. During oscillations, neglecting energy losses, the energy gets transferred between potential energy and kinetic energy. The natural frequency of oscillations, at which the pendulum would normally oscillate, depends on the value of gravity, length and so on. The friction in the medium would reduce the energy and cause the pendulum to slow down.

The same is true in an electric circuit. There is the energy stored in the electrostatic field in the capacitance and the energy stored in the electromagnetic field in the inductance. Oscillations occur when the energy gets transferred between these two forms. Resistance present in the circuit would cause energy losses and the resulting oscillations would decrease.

Series Resonance

Series resonance occurs in a circuit where the different energy storage elements are connected in series.

Consider the circuit shown in the figure.



At an angular frequency of ω, the value of the impedance is given by

$$Z = R + j\omega L + \frac{1}{j\omega C} = R + j(\omega L - \frac{1}{\omega C})$$

(i) Consider the value of ω at which the current magnitude becomes a maximum for a given voltage. This also corresponds to the impedance magnitude becoming a minimum, or the voltage becoming a minimum for a given current.

magnitude of impedance = | Z |, $\quad |Z|^2 = R^2 + (\omega L - \frac{1}{\omega C})^2$

phase angle of Z, $\quad \theta = \tan^{-1}\left[\frac{\omega L - \frac{1}{\omega C}}{R}\right]$

The condition for maximum or minimum impedance can either be obtained by differentiation of | Z | or even $|Z|^2$ or by inspection from physical considerations.

Since $|Z|^2$ consists of the sum of two square terms, the minimum value for any of the components would be zero. Since only the second term is dependant on ω, the minimum value could occur when the second term is zero.

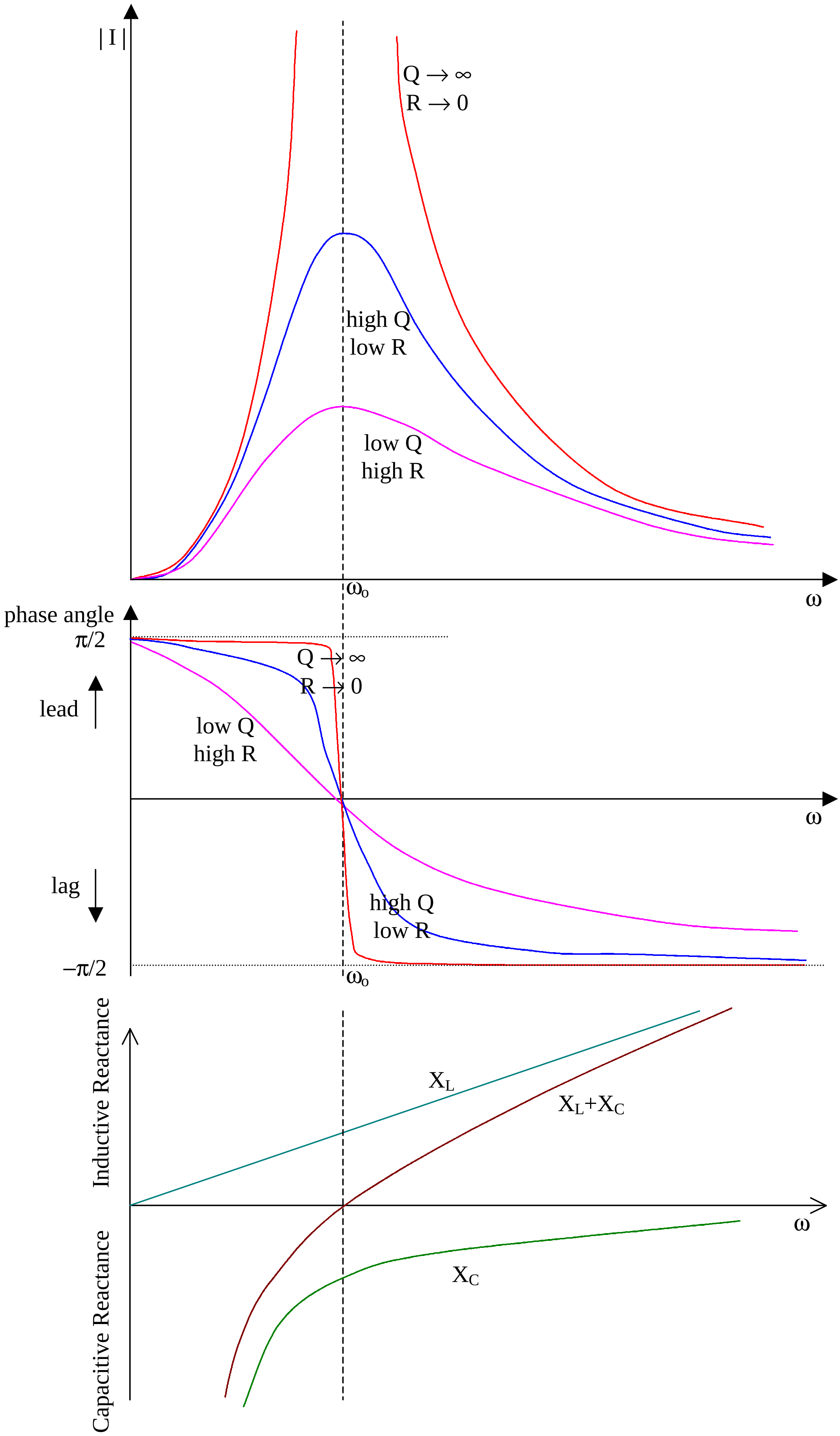
i.e. for minimum value of $|Z|^2$, $(\omega L - \frac{1}{\omega C}) = 0$ *or* $\omega L = \frac{1}{\omega C}$ *or* $\omega_o = \frac{1}{\sqrt{LC}}$

This would also correspond to the minimum value of Z.

If the current were considered, $|I| = \frac{E}{\left|R + j(\omega L - \frac{1}{\omega C})\right|} = \frac{E}{R^2 + (\omega L - \frac{1}{\omega C})^2}$

phase angle of I $= -\theta$

Let us now look at the variation of current magnitude and phase angle with frequency ω.



| I |
Q → ∞
R → 0
high Q
low R
low Q
high R
ω₀
ω
phase angle
π/2
lead
lag
Q → ∞
R → 0
low Q
high R
high Q
low R
−π/2
ω₀
ω
Inductive Reactance
Capacitive Reactance
X_L
X_L+X_C
X_C
ω

Now consider the frequency at which the power factor of the circuit becomes unity.

This occurs when the imaginary part of the impedance, or the current becomes zero.

i.e. $\quad j\,(\omega L - \dfrac{1}{\omega C}) = 0$, or at $\omega = \omega_o$. (which is the same condition as before).

It is seen from the plots of the magnitude of current and the phase angle that the shape changes dependant on the value of the series resistance of the circuit. It is seen that when the series resistance tends to zero, near perfect resonance occurs, giving a current magnitude tending to infinity. Similarly, as the series resistance tends to zero, the angle of the current with respect to the voltage tends to either $\pi/2$ or $-\pi/2$, changing at $\omega = \omega_o$.

The series resistance r in a series circuit, defines the quality of the resonance. If the resistance is low, the power loss is low and hence the quality is high. Also, if we compare it with the value of the impedance of either the inductance or capacitance at resonance, low r means that r is much less than either $L\omega_o$ or $1/C\omega_o$.

Quality Factor

Thus in the simplest terms, we could defined the quality of the circuit in terms of either the ratio of $L\omega_o$ to r or $1/C\omega_o$ to r.

Thus Quality = $\dfrac{L\omega_o}{r} = \dfrac{1}{C\omega_o r}$ at resonance, for a series circuit

[For a parallel circuit, the quality would actually increase when the value of resistance R tends to infinity rather than zero, and hence the inverse ratio would define the quality]

Thus we have a means of defining the quality of a resonant circuit, but which would appear to cause confusing results if the series and parallel circuits are considered.

Thus we would like to have a definition for the ***quality factor Q*** of a circuit in terms of quantities which do not vary dependant on the nature of the circuit.

The quality of the resonant circuit is in fact a relationship between the maximum energy stored in the energy storing elements (L or C) and the energy dissipation in the resistive elements (r or R) in the circuit.

Since the quality of a circuit was originally defined in terms of the ratio of impedance earlier, we would like to obtain the same answers for the simple series and parallel circuits with the definition based on energy. Thus the ***Q*** factor has a multiplying constant $\boldsymbol{2\pi}$ associated with it, in addition to the ratio of energies as follows.

$$Q = 2\pi \cdot \frac{\text{maximum stored energy}}{\text{energy dissipation per cycle}}$$

We will now verify whether this gives the correct answer for the simple series circuit considered earlier.

[For comparison purposes, consider the case of a simple pendulum. Note that the maximum kinetic energy occurs at the minimum point, and the maximum potential energy occurs at the highest point and that the stored energy does not change if friction were absent. However it is easier to calculate either the maximum kinetic energy only or the maximum potential energy only, rather than calculate both]

In a similar manner, the maximum energy stored in the electromagnetic field occurs in the inductor when the current is a maximum, and the maximum energy stored in the electrostatic field occurs in the capacitor when the voltage is a maximum.

The total energy does not change unless dissipated by the resistive elements in the circuit. However for calculation purposes, it is easier to either consider the energy stored at either the peak of the current or the peak of the voltage.

For a series circuit, it is easier to talk about the current through the capacitance rather than the voltage across the capacitor.

Thus for a current of $i(t) = I_m \sin \omega t$

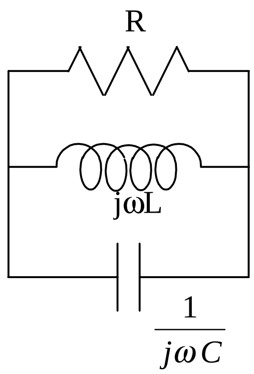
Maximum energy stored in the inductor in a series circuit = $\frac{1}{2} L I_m^2$

Energy dissipated per cycle in the series resistor = $r I_{rms}^2 T$, where $T = 2\pi/\omega$

Therefore the Q-factor for the series circuit would be given b

$$Q = \frac{\frac{1}{2} L I_m^2}{r I_{rms}^2 T} = 2\pi . \frac{\frac{1}{2} L I_m^2}{r(I_m/\sqrt{2})^2 \cdot 2\pi/\omega} = \frac{L\omega}{r}$$ which is the expected result.

Parallel Resonance



Parallel resonance occurs in a circuit where the different energy storage elements are connected in parallel.

Consider the circuit shown in the figure.

At an angular frequency of ω, the value of the admittance is given by

$$Y = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C = j\frac{1}{R} + j(\omega C - \frac{1}{\omega L})$$

magnitude of admittance = | Y |, $\quad |Y|^2 = \frac{1}{R^2} + (\omega C - \frac{1}{\omega L})^2$

parallel resonance is seen to occur when the imaginary part is zero. i.e. $\frac{1}{\omega L} = \omega C$

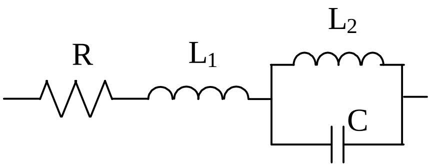
It is also seen that this also corresponds to unity power factor.

In practical circuits, series resonance and parallel resonance will occur in different parts of the same circuit. The unity power factor resonance may correspond to one of these.

Example 1

Find the types of resonance and the resonance frequencies of the circuit shown in the figure.

Solution



$$Z = R + j\omega L_1 + \frac{j\omega L_2 \cdot \frac{1}{j\omega C}}{j\omega L_2 + \frac{1}{j\omega C}}$$

$$= R + j\omega L_1 + \frac{j\omega L_2}{1 - \omega^2 L_2 C}$$

Consider each type of resonance condition in turn

(a) when the power factor is unity

the equivalent impedance is purely real. Therefore $\omega L_1 + \frac{\omega L_2}{1 - \omega^2 L_2 C} = 0$

i.e. $L_1 + L_2 - \omega^2 L_1 L_2 C = 0$, or $\omega^2 = \dfrac{L_1 + L_2}{L_1 L_2 C} = \dfrac{1}{L_{eq} C}$, where $L_{eq} = \dfrac{L_1 L_2}{L_1 + L_2}$

It is seen that the value of equivalent inductance obtained is the parallel equivalent of the two inductances of the circuit.

Thus the unity power factor resonance frequency corresponds to $\omega = \dfrac{1}{\sqrt{L_{eq} C}}$

(b) when the impedance of the circuit is a minimum

Examination will show that this also occurs when in fact corresponds to a minimum value of impedance or series resonance. But this need not be the case for all examples.

(c) when the impedance of the circuit is a maximum

$$| Z |^2 = R^2 + \left[\omega L_1 + \frac{\omega L_2}{1 - \omega L_2 C}\right]^2$$

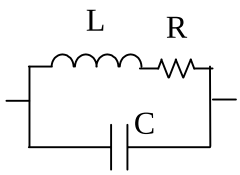
This impedance will have a maximum value of infinity at $(1 - \omega L_2 C) = 0$

$\therefore$ resonant frequency for parallel resonance = $\dfrac{1}{\sqrt{L_2 C}}$

Example 2

Find the unity power factor resonance frequency of the circuit shown in the figure. Also determine the parallel resonance frequency if R = 20 Ω, L = 10 mH, and C = 4 μF.

Solution



$$Z = \frac{(R + j\omega L)\cdot\dfrac{1}{j\omega C}}{R + j\omega L + \dfrac{1}{j\omega C}} = \frac{R + j\omega L}{(1 - \omega^2 LC) + j\omega CR}$$

for unity power factor resonance, the impedance Z is purely real.

$$\therefore \quad \frac{R}{(1 - \omega^2 LC)} = \frac{\omega L}{\omega\, CR}, \quad \text{i.e. } C R^2 = L (1 - \omega^2 L C)$$

$$\omega^2 = \frac{L - CR^2}{L^2 C}, \quad \text{or} \quad \omega = \sqrt{\frac{L - CR^2}{L^2 C}}$$

Unity power factor resonance $\omega = \sqrt{\dfrac{10\times10^{-3} - 4\times10^{-6}\times400}{100\times10^{-6}\times4\times10^{-6}}} = 4582.6\ rad/s = \quad 729.3\ Hz$

Now consider the magnitude of the impedance

$$| Z |^2 = \frac{R^2 + \omega^2 L^2}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$

for maximum value of | Z |, $| Z |^2$ must be a maximum. i.e. $\dfrac{d\,| Z |^2}{d\omega}$

i.e. $[(1-\omega^2LC)^2 + \omega^2C^2R^2].\cancel{2\omega}L^2 - (R^2 + \omega^2L^2)[2(1-\omega^2LC).(-\cancel{2\omega}LC) + \cancel{2\omega}C^2R^2] = 0$

i.e. $(1-\omega^2LC)^2L^2 + \omega^2C^2L^2R^2 + 2R^2\,LC(1-\omega^2LC) + 2\,\omega^2L^2\,LC(1-\omega^2LC) - (R^2 + \omega^2L^2)C^2R^2 = 0$

$-\omega^4L^4\,C^2 + \omega^2\,[\cancel{-2L^3C} + \cancel{C^2L^2R^2} - 2R^2\,L^2C^2 + \cancel{2\,L^3C} - \cancel{L^2C^2R^2}] + L^2 + 2R^2\,LC - C^2R^4 = 0$

substituting values for the components

$$-0.16\times10^{-18}\,\omega^4 - 1.28\times10^{-12}\,\omega^2 + 1.2944\times10^{-4} = 0$$

$$\omega^4 + 8\times10^6\,\omega^2 - 0.8090\times10^{15} = 0$$

which is a quadratic equation in ω^2 which can be solved.

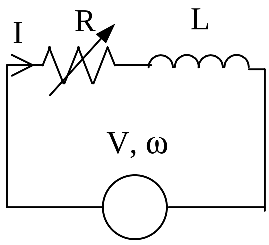
$$\omega^2 = -4\times10^6 \pm \sqrt{(16\times10^{12} + 0.8090\times10^{15})} = -4\times10^6 \pm 28.723\times10^6 = 26.723\times10^6$$

$$\omega = 4972 \text{ rad/s} = 791.4 \text{ Hz}$$

You will notice that the resonance frequency is close but not the same as the earlier result.

Loci Diagrams for RL and RC circuits

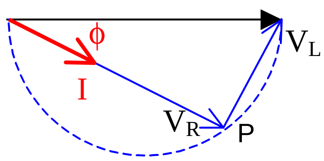
(a) Series RL circuit



Consider the series RL circuit, with a constant voltage V at constant frequency ω applied across it.

If the resistance R of the circuit varies, the component voltages V_R and V_L would vary keeping the complex sum a constant. This can be plotted in the following manner

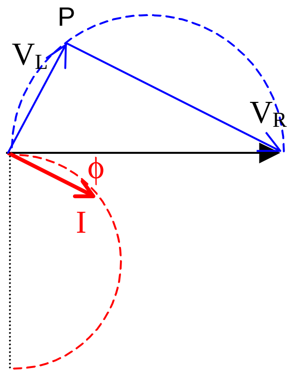
$$V = (R + j\,\omega L)\,.I \qquad = (R + j\,X)\,.I = V_R + V_L$$



If the applied voltage V is taken as reference, the current I would be lagging the voltage by a phase angle ϕ.

The voltage V_R is in phase with the current I and the voltage V_L would be quadrature leading the current I (or current I lagging the voltage V_L by 90^o).

Since V_R and V_L must be mutually perpendicular, when R varies, the point P must move along a semi-circle. This semi-circle is the locus of the point P as R is varied.



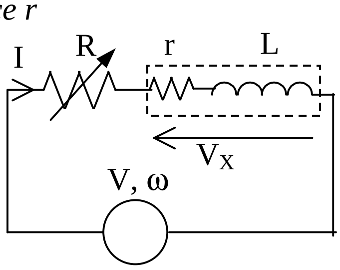
Let us now look at the variation of I as R varies and X is kept fixed.

This is best understood by considering the addition as $V = V_L + V_R$.

This has the phasor diagram as shown. V_L is first drawn perpendicular to I (such that the current lags the voltage by 90^o). The locus of the point P will again be a semi-circle.

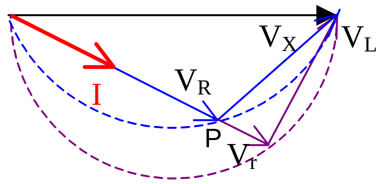
Since X is fixed, there will be a definite proportion between the length of phasor I and the length of phase V_L. Also, I is always lagging the voltage V_L by 90^o. Thus the locus of I must also be a semi-circle lagging the semi-circle for voltage V_L by 90^o as shown.

(b) Series RL circuit with the inductance having a finite resistance r



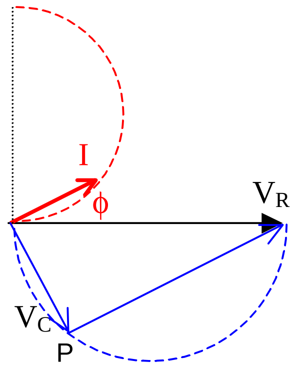
Practical inductances would usually have a significant value of winding resistance r which may not be neglected. In such a case, the measured voltage V_X across the practical inductor would have two components corresponding to the inductive component and the resistive component respectively.

Let us consider the phasor diagram for the voltages.



The locus of the node between the resistor and the inductor is no longer a semi-circle. However, when the total resistance of the circuit, $R + r$ is considered, the locus of that point with the pure inductance part remains a semi-circle. If X is the variable, then the ratio of V_R to V_r remains a constant and the value of the internal resistance of the coil can be determined.

(c) *Series RC circuit*



The series RC circuit is analysed in a similar manner to the series RL circuit. However the current would be leading in this case instead of lagging the voltage.

Unlike in the case of the practical inductor, the practical capacitor does not have significant amount of losses. Thus the practical locus diagram can be considered to be the same as the theoretical diagram.

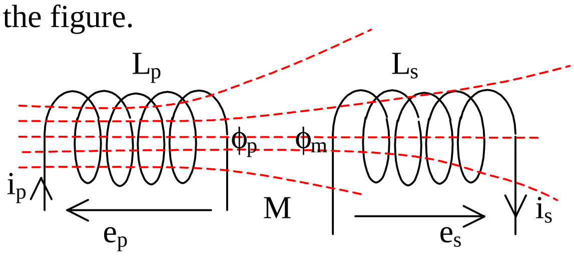
(d) *Parallel RL and RC Circuits*

The loci diagrams of parallel RL and RC circuits may be obtained in a manner similar to obtaining the loci diagrams for the series circuits.

Mutual Inductance

Mutual coupling between coils exist, when one coil is in the magnetic field created by another coil.

Consider two magnetically coupled coils as shown in the figure.



When a varying current $i_p(t)$ flows in the primary winding, then a varying flux ϕ_p is produced in the same coil and produces a back emf $e_p(t)$. Part of the flux produced ϕ_m can link with a second coil. Since this flux will also be varying, an induced emf e_s will be produced in the second coil.

$$e_p = N_p \frac{d\phi_p}{dt}, \qquad e_s = N_s \frac{d\phi_m}{dt}, \qquad \phi_m = k \, . \, \phi_p$$

The mutual flux ϕ_m will be directly proportional to the primary flux ϕ_p, with a coefficient of coupling k slightly less but very close to unity.

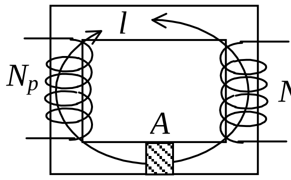
In the linear region of the magnetisation characteristic, the flux produced ϕ_p will be proportional to the current i_p producing the flux. Thus the mutual flux ϕ_m will also be proportional to the current i_p. Thus the induced emf e_s in the secondary will be proportional to the rate of change of current in the primary, with the constant of proportionality being defined as the Mutual inductance M_{sp}.

i.e. $$e_s \propto \frac{di_p}{dt} = M_{sp}\frac{di_p}{dt}, \qquad \text{where} \quad M_{sp} = \frac{N_s\phi_m}{i_p} = \frac{kN_s\phi_p}{i_p}$$

Thus the (coefficient of) Mutual inductance is defined as the flux linkage produced in a secondary winding per unit current in the primary winding.

The sign associated with the mutual inductance can be positive or negative dependent on the relative directions of the two windings. In other words, the direction of the voltage induced in the second coil will depend on the relative direction of winding of the two coils.

The flux ϕ_p in a coil is be related to the current i_p producing the flux through the self inductance of the coil L_p and may be expressed in terms of the dimensions of the magnetic path (length l and cross-section A) as follows.



i.e. $L_p = \dfrac{N_p \phi_p}{i_p} = \dfrac{N_p B_p A}{i_p} = \dfrac{N_p \mu H_p A}{i_p}$, but $N_p i_p = H_p l$

$\therefore \quad L_p = \dfrac{N_p \mu N_p i_p A}{i_p \cdot l} = \dfrac{N_p^2 \mu A}{l}$

In like manner, the mutual inductance may be derived in terms of the dimensions.

$$M_{sp} = \frac{k_{sp} N_s \phi_p}{i_p} = \frac{k_{sp} N_s B_p A}{i_p} = \frac{k_{sp} N_p \mu H_p A}{i_p} = \frac{k_{sp} N_s \mu N_p i_p A}{i_p \cdot l} = \frac{k_{sp} N_p N_s \mu A}{l}$$

similarly, $M_{ps} = \dfrac{k_{ps} N_s N_p \mu A}{l}$

The coupling between the primary and the secondary, for all practical purposes, will be identical to the coupling between the secondary and the primary, so that $k_{ps} = k_{sp}$.

Thus it can be seen that for all practical purposes, the mutual inductance between the primary winding and the secondary winding is identical to the mutual inductance between the secondary winding and the primary winding, and would usually be denoted by a single quantity M and a single coefficient k.

$$M = \frac{k N_s N_p \mu A}{l}, \quad L_p = \frac{N_p^2 \mu A}{l}, \quad L_s = \frac{N_s^2 \mu A}{l}$$

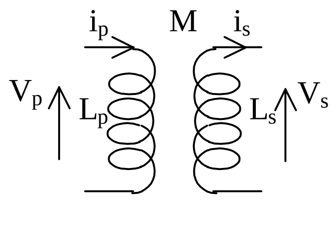
giving $M^2 = k^2 L_p L_s$ or $M = k\sqrt{L_p L_s}$

Energy stored in a pair of mutually coupled coils

We know that an inductor stores energy in the electromagnetic field equal to $\tfrac{1}{2} L I^2$.

In like manner, when there is mutual coupling present, the total energy stored by two coils is different from the addition of the $\tfrac{1}{2} L I^2$ terms. This change is the effective energy stored in the mutual inductance.

Consider the pair of coils shown.



Total energy stored $= \int v_p\, i_p\, dt + \int v_s\, i_s\, dt$

$$= \int (L_p \frac{d\,i_p}{dt} \pm M \frac{d\,i_s}{dt}).i_p dt + \int (L_s \frac{d\,i_s}{dt} \pm M \frac{d\,i_p}{dt}).i_s dt$$

$$= \int L_p\, i_p\, di_p \pm \int M\, i_p\, di_s + \int L_s\, i_s\, di_s \pm \int M\, i_s\, di_p$$

This may be grouped to give

Total energy stored $= \int L_p\, i_p\, di_p + \int L_s\, i_s\, di_s \pm \int M\, i_p\, di_s \pm \int M\, i_s\, di_p$

$$= \tfrac{1}{2} L_p i_p^{\,2} + \tfrac{1}{2} L_s i_s^{\,2} \pm M\, i_p i_s$$

Therefore the effective energy stored in the mutual inductance corresponds to $\pm M\, i_p\, i_s$.

Equivalent inductance of 2 mutually coupled coils in series

It will be seen that the coils can either be connected so that the fluxes aid each other, as in the first figure and the fluxes oppose each other as in the second figure.

Since each current is i,

Total energy stored = $\tfrac{1}{2} L_1 i^2 + \tfrac{1}{2} L_2 i^2 + M\, i.i$ or $\tfrac{1}{2} L_1 i^2 + \tfrac{1}{2} L_2 i^2 - M\, i.i$

If a single equivalent inductor L_{eq} is considered, the total energy stored would be $\tfrac{1}{2} L_{eq} i^2$.

Thus equating the energies we have

$$\tfrac{1}{2} L_{eq} i^2 = \tfrac{1}{2} L_1 i^2 + \tfrac{1}{2} L_2 i^2 + M\, i.i = \tfrac{1}{2} L_1 i^2 + \tfrac{1}{2} L_2 i^2 - M\, i.i$$

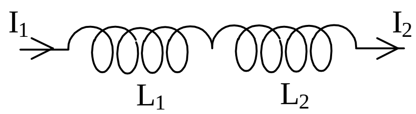
i.e. $L_{eq} = L_1 + L_2 + 2M \qquad = L_1 + L_2 - 2M$

Thus it is seen that the effective inductance can either increase or decrease due to mutual coupling dependant on whether the coils are wound in the same direction or not.

Example 3

Consider a couple of coils connected in series as shown in the figure.

Let each coil have N turns, so that the total series connected coil has 2N turns.



If the dimensions of the common magnetic circuit on which these are wound have area A and length l

$$L_1 = L_2 = \frac{N^2 \mu A}{l}, \text{ and the total coil } L = \frac{(2N)^2 \mu A}{l} = \frac{4N^2 \mu A}{l}.$$

$L_1 + L_2$ is obviously not equal to this total. What went wrong ?

If the two coils are wound on the same magnetic circuit, very closely, then there would be mutual coupling with the coefficient of coupling almost unity.

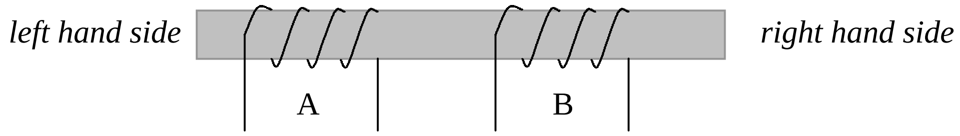
Then, $M = k\sqrt{L_p L_s} = 1 \times \sqrt{\frac{N^2 \mu A}{l} \cdot \frac{N^2 \mu A}{l}} = \frac{N^2 \mu A}{l}$

Thus the total inductance would be $L_1 + L_2 + 2M = \dfrac{4N^2 \mu A}{l}$ as expected.

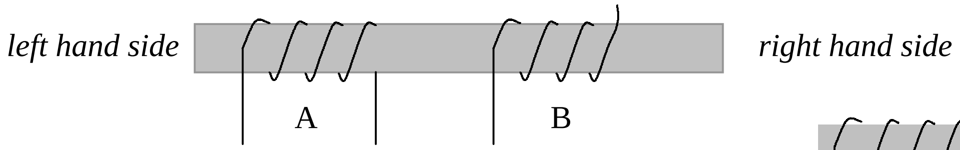
Magnetic circuit Analysis

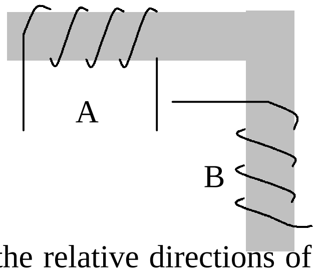
As you are aware, when a coil is wound round a magnetic core, the core becomes magnetised and one side becomes a north pole and the other side becomes a south pole. There are different methods of remembering which side is which. One of the simplest methods to remember is given below.

In this method, if we look at the coil from one side, if the current direction is anti-clockwise the nearer side is a north pole (which is also seen from the arrow direction of N); and if the current direction is clockwise, the nearer side is a south pole (which is also seen from the arrow direction of S).



If you look at coil A and coil B, we can easily visualise that they are both wound in the same direction. That is, if we examine the coil from the left hand side when a current is entering the left hand side end of the coil, each coil would produce a south pole at the left hand side and a north pole at the right hand side.





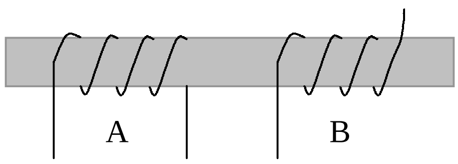
With the slightly differently drawn diagram it would be less obvious.

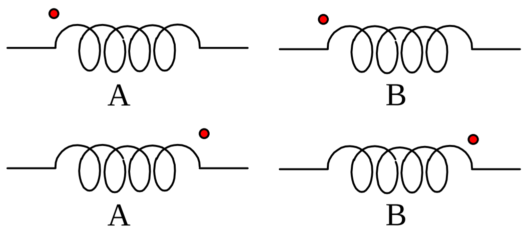
Further if the magnetic circuit was not a straight line, confusion could tend to enter the decision. A method would be to open out the bent path to make it a straight line and then compare directions.

Each time we analyse a magnetic circuit, we would need to look at the relative directions of the two coils. However, once wound, the relative directions of the coils would not change, independent of how we look at the coils. Thus we can use a simpler method to know the relative directions of the coils. For this purpose we use ***dots*** to denote similar ends of different coils.

Dot Notation

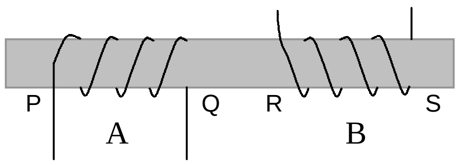
The relative directions of coils is important in determining the direction of induced voltage with mutual coupling. Consider the following magnetic circuits, and the corresponding electrical circuit with polarities defined by the dotted ends.

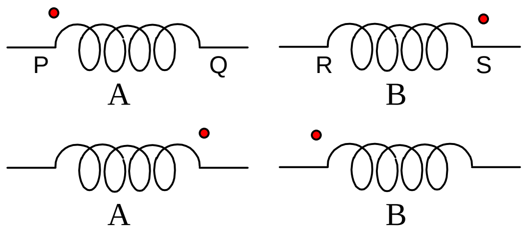




Note that there are two possible ways of drawing the dotted ends, but both give the same relationship to each other. The dots do not indicate that one end is a north pole or a south pole, as this would depend on which direction the current passes in the coil. The dots indicate similar ends in that if a changing flux passes in the magnetic core, then the induced voltages would either all correspond to direction of the dot or all to the opposite direction. Thus they indicate similar ends of windings only.

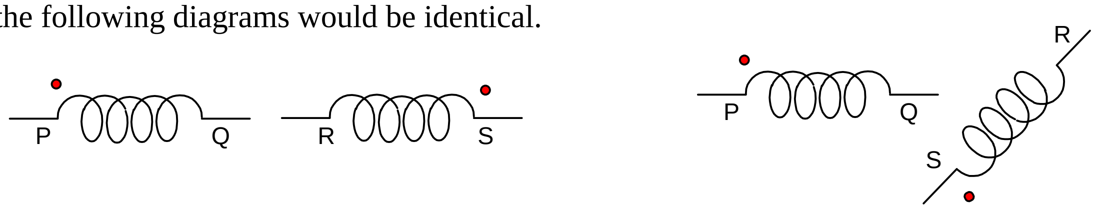
Consider the same core as before, but with the coil B wound in the opposite direction.





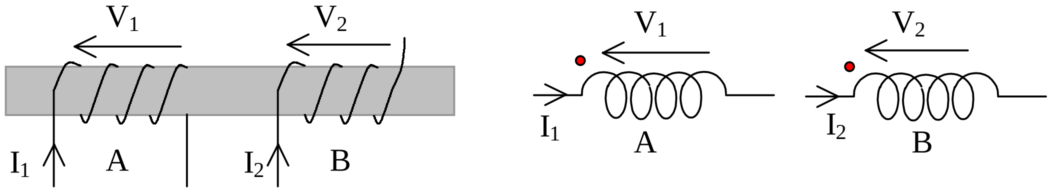
The position of the dots again indicate the different winding directions.

Once the dots are drawn to indicate similar ends, unlike in the magnetic circuit, there is no real necessity to physically place the winding diagram in the same physical position. Thus the following diagrams would be identical.



Thus it is seen that once the dots have been marked to identify similar ends, the physical positions have no meaning and we can draw them where it is convenient.

Let us now see how, once the dots have been marked, in an electrical circuit the correct directions of the induced voltage scan be obtained.



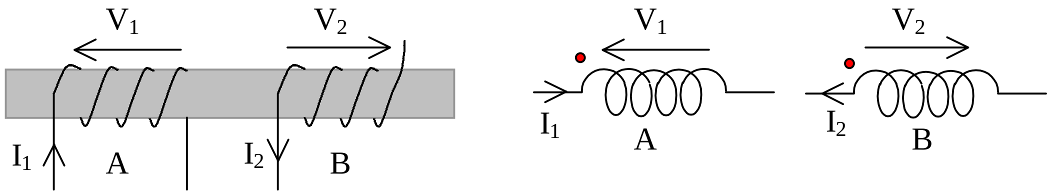
If a single coil is considered, the induced voltage would always be opposite to the direction of current flow in that winding. The voltages V_1 and V_2 have been drawn as such in the above diagrams.

Let us now see what would happen due to each individual current.

Since each winding is wound in the same direction, the current I_1 would induce voltages in A and B in the same sense. Also if the current I_2 is marked as shown, then it too would induce voltages in A and B in the same sense, and also in the same sense as due to the current I_1. The fluxes and the corresponding induced emfs are thus additive as the coils are wound in the same direction. This is also shown by the non magnetic equivalent circuit.

In other words, the voltage drop due to the self inductance term and the voltage drop due to the mutual inductance term have the same sign.

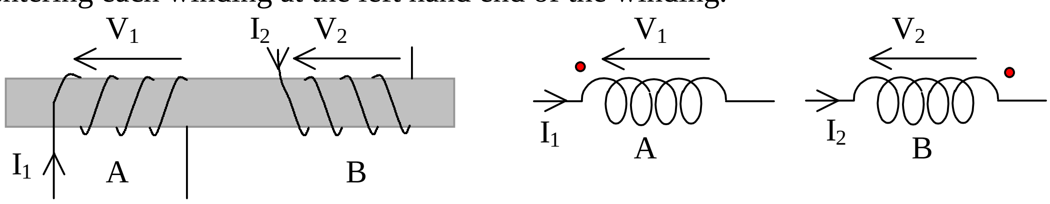
Let us now see what would happen if the direction of the current I_2 is changed without changing any physical considerations.



Obviously, the direction of the voltage V_2 due to this current I_2 on its own winding will change in direction. However, the induced voltage due to the current I_1 will have the same direction for both coils, as the physical directions of the two coils are still the same as before.

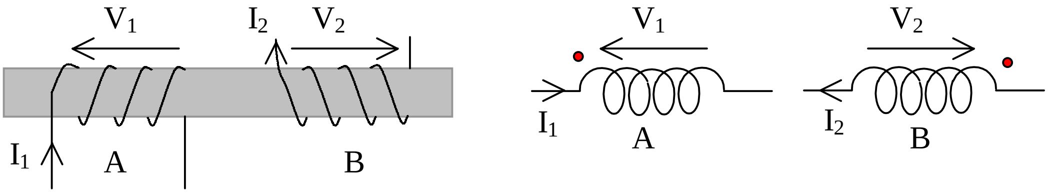
Thus it is seen that the voltage drop due to the self inductance term and the voltage drop due to the mutual inductance term have opposite signs.

Let us see what would happen if the direction of one of the coils were changed but with the currents entering each winding at the left hand end of the winding.



If the direction of one of the windings is changed, the mutual induced voltage will change in direction relative to the self inductance term. Thus again the voltage drop due to the self inductance term and the voltage drop due to the mutual inductance term have opposite signs.

The final possibility is if both the direction of the winding and the current direction are changed.



It will easily be seen that the voltage drop due to the self inductance terms and that due to the mutual inductance term must have the same sign.

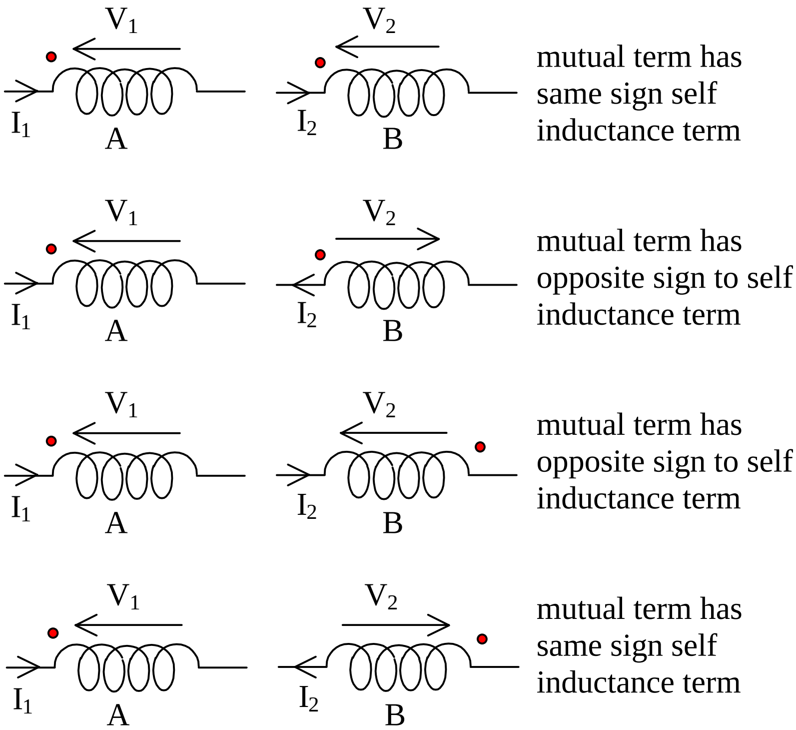
The above derivations were done primarily based on the magnetic circuits appearing in the above 4 cases.

Let us now see what has happened with the electrical circuits.

The summary of the 4 cases are shown. It is seen that in cases 1 and 4, the mutual inductance term has the same sign, while in cases 2 and 3 the mutual inductance term has opposite sign to the self inductance term.

Let us see what properties actually cause the above situation.

It is seen that in both cases 1 and 4, the currents marked enter the coil at the dotted end, while in both cases 2 and 3, one current enters at a dotted end, while the other current leaves at the dotted end.



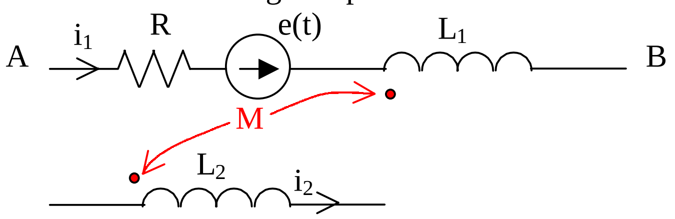
The above results can be stated in the following manner.

- *If both currents either enter at the dotted end, or both currents leave at the dotted end, the sign associated with the mutual inductance is positive, and the mutual inductance term in the voltage drop would have the same sign as the self inductance term.*

- *If one current enters at a dotted end and the other current leaves at the dotted end, the sign associated with the mutual inductance is negative, and the mutual inductance term in the voltage drop would have the opposite sign to the self inductance term.*

Example 4

For the circuit shown, write down the voltage drop across AB.



V_{AB} = R. i_1 – e(t) + (L p i_1 + M p i_2)   where p = d/dt

V_{BA} = – R. i_1 + e(t) – (L p i_1 + M p i_2)

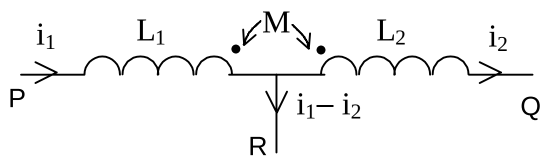
It is seen that since both currents enter at the respective dotted ends, the mutual inductance term and the self inductance term has the same sign. This is true even if the voltage is measured in the opposite direction. Thus it is usually advisable to treat the self inductance terms and the mutual inductance terms together within parenthesis to avoid wrong negation.

In the case of steady state a.c. analysis, the input quantities of sinusoidal with frequency ω, so that the differential would give a multiplication of ω and a phase shift of π/2. Thus in this case, p = jω would give the required equations, so that the impedance due to mutual inductance for a.c. would in general be ± jω M.

Non-coupled Equivalent circuit of simple coupled circuits

When mutual coupling terms are present, voltage drops in a particular branch does not depend only on the currents in that branch. Thus using the equations for series and parallel equivalent of branch circuits cannot be implemented. This problem can be sometimes avoided by obtaining a non-coupled equivalent circuit.

(a) *coupled coils being on two arms of a T-junction*



Consider the pair of mutually coupled coils shown. It is seen that the two coupled coils are on two arms of a T-junction with currents i_1 and i_2 flowing in them and a current (i_1 – i_2) flowing in the common branch.

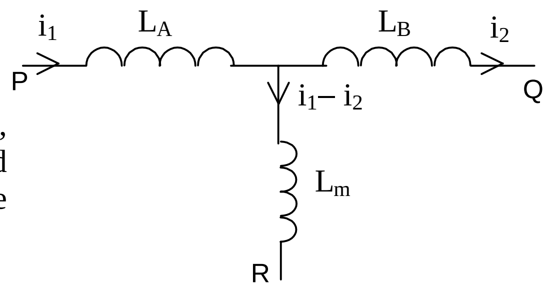
Kirchoff's current law has already been applied in marking the currents.

Applying Kirchoff's voltage law between PR, and then again RQ, we can write

$$V_{PR} = L_1 \frac{d\,i_1}{d\,t} - M\frac{d\,i_2}{d\,t} \qquad \text{or} \qquad L_1\,p\,i_1 - M\,p\,i_2$$

or V_{PR} = $j\omega L_1 i_1 - j\omega M i_2$ with sinusoidal alternating current

Note that current i_1 leaves the dotted end of L_1, while the current i_2 enters the dotted end of L_2, so that the sign associated with the mutual inductance is *negative*, or stated in other words, the voltage drop term due to the mutual inductance has the opposite sign to that due to the self inductance term across the corresponding element.



and V_{RQ} = L_2 p i_2 - M p i_1

If a non-coupled equivalent circuit is to be obtained, voltage drops in a branch should only correspond to currents in it own branch. So we will re-write the 2 equations as follows to achieve this.

$$V_{PR} = L_1\, p\, i_1 - M\, p\, i_2 - M\, p\, i_1 + M\, p\, i_1$$

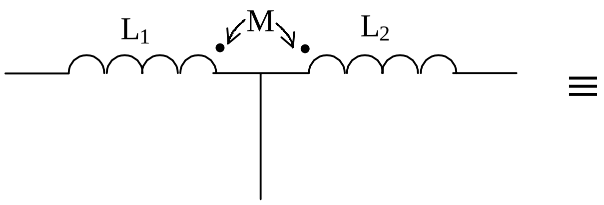
i.e. $$V_{PR} = (L_1 - M)\, p\, i_1 + M\, p\,(i_1 - i_2)$$

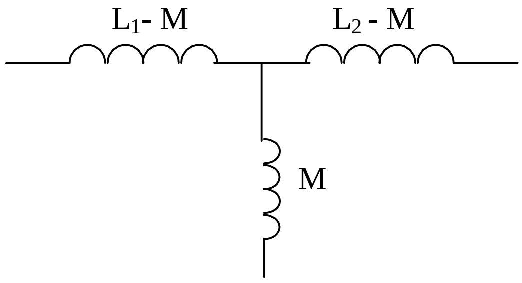
similarly

$$V_{RQ} = (L_2 - M)\, p\, i_2 + M\, p\,(i_1 - i_2)$$

These two equations would be satisfied with $L_A = L_1 - M$, $L_B = L_2 - M$, and $L_m = M$

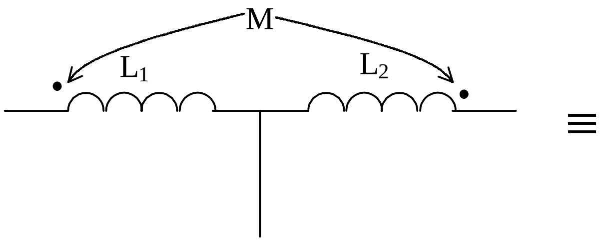
This transformation will be valid, independent of what the directions marked for the currents in the diagrams. [This is like, we have to mark current directions when proving the expression the parallel equivalent of two resistors, but the result derived is independent of the original markings]. Thus

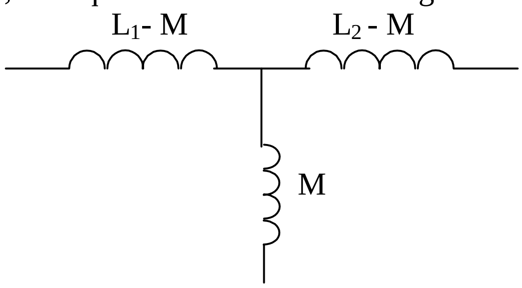




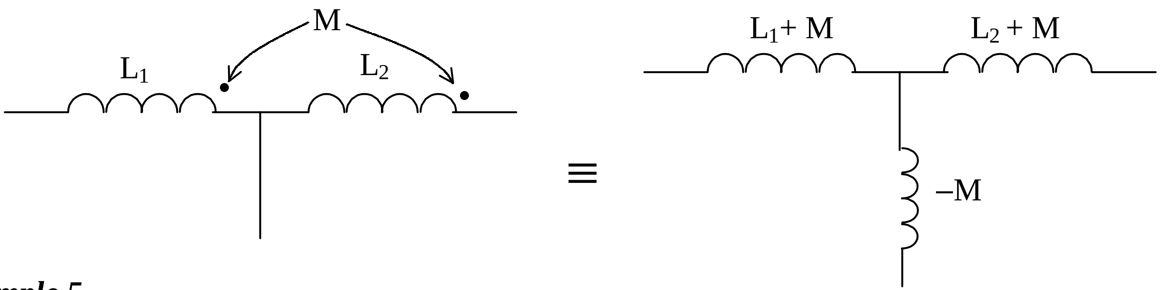
Note that L – M appears when the two coils are opposing each other. In this case the common branch has an added +M appearing on it.

In like manner, if the two dots were at the further ends, the equations would be unchanged and the equivalent circuit would also be unchanged.



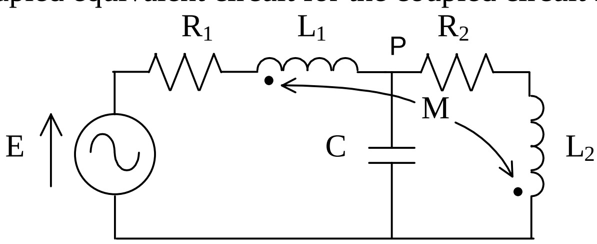


Also, if the position of one of the dots is changed, then the two coils would be aiding each other, and the terms can be shown to correspond to L + M with a common term of – M.



Example 5

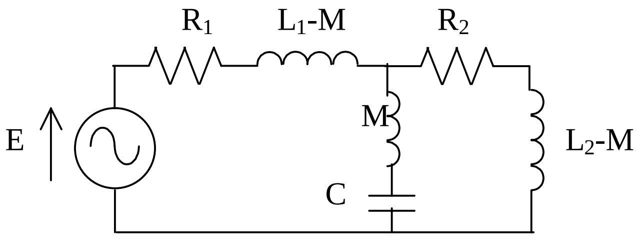
Write down the non-coupled equivalent circuit for the coupled circuit shown.



Solution

In order to get the correct sign in the non-coupled equivalent circuit, consider an imaginary current to be flowing through coil 1 and coil 2, forgetting about all other elements. This will tell us whether the two coils are aiding or in opposition. In this case it is seen that they are opposing. [In fact you will notice that the two coils do not exactly meet at a common node, but that the node P would have that property if the positions of R_2 and L_2 were interchanged. Since they are series elements, the voltage drop equation would not change even if the order were changed. [However, the intermediate point of connection between these elements would have a different voltage from earlier, as the drops are occurring in a different order.]

The non-coupled equivalent circuit may thus be drawn as follows.

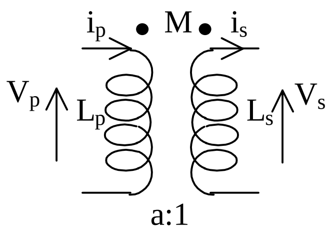


This circuit no longer has mutual coupled elements, but the elements have taken into account the affects of mutual inductance.

Thus the problem may be solved as for any alternating current problems.

Transformer as a pair of mutually coupled coils

The transformer is in fact a pair of mutually coupled coils. It can thus be analysed in that manner.



From Kirchoff's voltage law

$$V_p = L_p\, p\, i_p - M\, p\, i_s$$

and $$V_s = -(L_s\, p\, i_s - M\, p\, i_p)$$

From properties of ideal transformers we know that the ratio of V_p to V_s is close to a and i_p to i_s is close to $1/a$. Thus V_p and aV_s are comparable quantities while i_p and i_s/a are comparable.

Thus the equations will be re-written in terms of these comparable variables as follows.

$$V_p = L_p\, p\, i_p - aM\, p\, i_s/a$$

and $$aV_s = -(a^2L_s\, p\, i_s/a - aM\, p\, i_p)$$

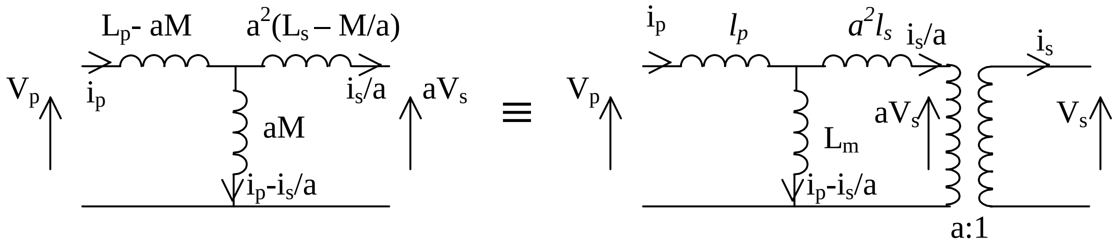
If the primary current and the primary equivalent of the secondary current are considered, the difference must correspond to the current leakage. Thus we may re-write in terms of this leakage current as

$$V_p = L_p\, p\, i_p - aM\, p\, i_s/a + aM\, p\, i_p - aM\, p\, i_p = (L_p - aM)\, p\, i_p + aM\, p\, .(i_p - i_s/a)$$

and $$aV_s = -(a^2L_s\, p\, i_s/a - aM\, p\, i_p) + aM\, p\, i_s/a - aM\, p\, i_s/a = -\, a^2(L_s - \frac{M}{a})\, p\, i_s/a + aM\, p.(i_p - i_s/a)$$

It will be noticed that the equations have been expressed either in terms of i_p and $i_p - i_s/a$ or in terms of i_s/a and $i_p - i_s/a$.

This allows a non-coupled equivalent circuit to be formed as follows which satisfy the modified equations.



This may also be drawn with an ideal transformer added so that the secondary quantities would become the unmodified voltage and current in the secondary side.

Also, M = k $\sqrt{L_p L_s}$, $L_p/L_s = a^2$ so that M = k $\sqrt{L_p \dfrac{L_p}{a^2}}$

giving aM = k L_p, L_p-aM = (1-k) L_p, $a^2(L_s – M/a) = a^2(1\text{-}k)\, L_s$

Since k is the coefficient of coupling, (1-k) corresponds to the leakage. Thus (1-k)L_p and (1-k)L_s correspond to the leakage inductances l_p and l_s of the primary and secondary windings. The shunt inductance *aM* corresponds to the magnetisation inductance L_m of the transformer. The transformer equivalent circuit is normally drawn with these variables rather than the self and mutual inductances.

Practical Transformer

The practical transformer, in addition to the above has resistances r_p and r_s in each of the primary and secondary windings. In addition, there is a loss (eddy current loss and hysteresis) in the core of the transformer based on the magnetic flux in the core (or the corresponding voltage applied to the windings). The winding resistances can easily be included in the primary and secondary sides (secondary resistance may be referred to the primary side and drawn as $a^2\, r_s$). The core loss is represented by a resistance R_c which appears in shunt across the magnetisation inductance. Thus the complete equivalent circuit of a practical transformer is given as follows.