Analysis of Non-Sinusoidal Waveforms – Part 2 – Laplace Transform

In the earlier section, we learnt that the Fourier Series may be written in complex form as

$$f(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_o t}$$

where the Fourier coefficient C_n is given by

$$C_n = \frac{1}{T}\int_{t_o}^{t_o+T} f(t)\cdot e^{-jn\omega_o t}\cdot dt$$

In the symmetrical form, the Fourier series is written with $t_o = -T/2$.

The Fourier series is written for a periodic function with period T, and discrete frequency components are obtained for the waveform. We saw that the fundamental frequency ω_o is related to the period T by the expression $\omega_o = 2\pi/T$.

Now consider the following waveforms.

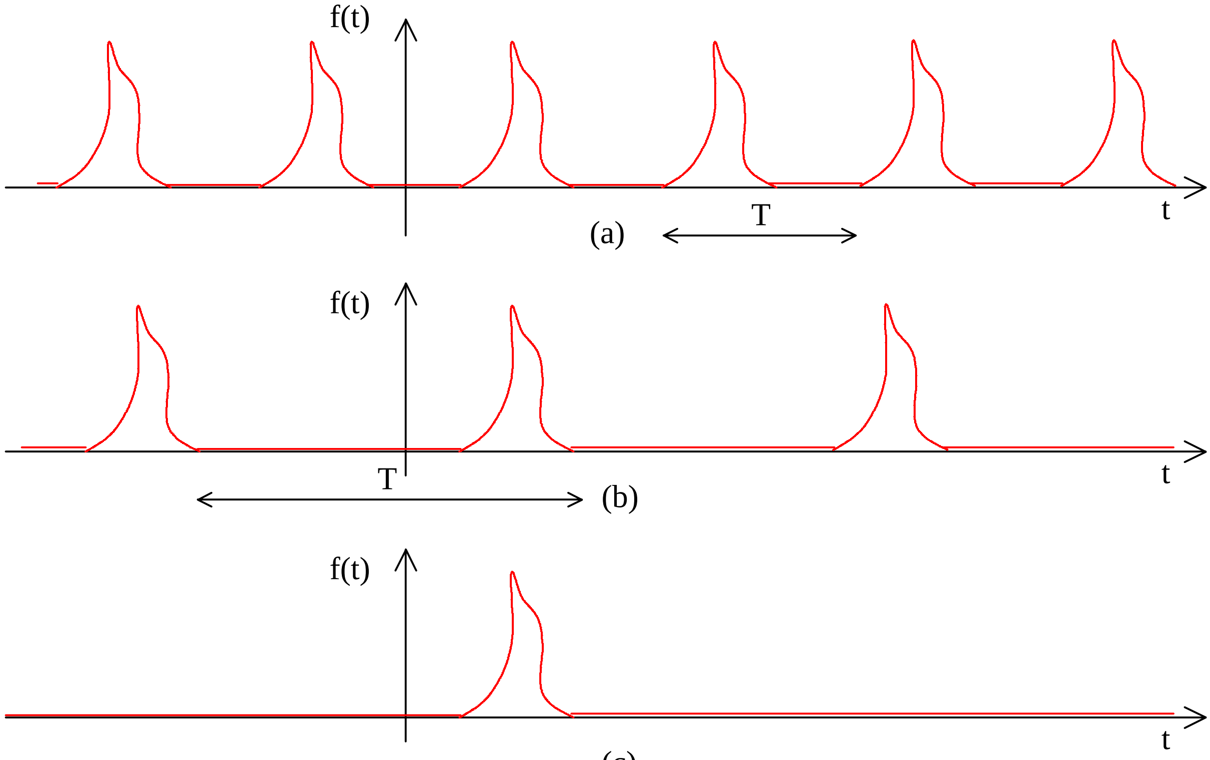


Figure 1 – Period of repetition gradually increased

In figure 1(a), the period of repetition is quite small and in (b) somewhat larger. Waveform (c) could be considered as one where the period of repetition has been increased up to infinity.

Thus any non-repetitive waveform may be considered as one which has a period T→ ∞, and the corresponding fundamental frequency $\omega_o = \frac{2\pi}{T} = \Delta\omega \to 0$.

It is also seen that the Fourier coefficient C_n in the symmetrical exponential series

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\cdot e^{-jn\omega_o t}\cdot dt \quad = \quad \Delta C \to 0$$

The frequencies involved are no longer discrete but continuous, so that the general frequency $n\omega_o$ corresponds to $\Sigma\, \Delta\omega \rightarrow \int d\omega = \omega$.

Thus for non-repetitive functions, the following can be written.

$$T \rightarrow \infty$$
$$\omega_o \rightarrow d\omega$$
$$C_n \rightarrow dC$$
$$n\omega_o \rightarrow \omega$$
$$\frac{1}{T} = f = \frac{\omega_o}{2\pi} \rightarrow \frac{d\,\omega}{2\pi}$$

Thus the expression for the complex Fourier Coefficient C_n becomes

$$dC = \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} f(t).e^{-j\omega t}.dt$$

dividing both sides by $d\omega$, this may be written as

$$F(\omega) = \frac{dC}{d\omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t).e^{-j\omega t}.dt$$ Definition of the **Fourier Transform**

The original function f(t) is now given as

$$f(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_o t} = \int_{-\infty}^{\infty} dC.e^{j\omega t}$$

from the definition, $dC = F(\omega).d\omega$, so that

$$f(t) = \int_{-\infty}^{\infty} F(\omega).e^{j\omega t}.d\omega$$ **Fourier Inverse Transform**

The Fourier Transform expression and the Fourier Inverse Transform expression together are known as the **Fourier Transform Pair.**

If we multiply the Fourier Transform by a constant and divide the Inverse Transform also by the same constant, we would again get a modified transform pair.

If we examine the two transform expressions, we see that they look very similar except that there is a difference of a negative sign in the exponent and a multiplying factor of 2π.

Thus we could define a symmetrical transform pair by using a factor of $\sqrt{2\pi}$.

In this case the **Symmetric Fourier Transform** is defined as

$$F_s(\omega) = \sqrt{2\pi}\,\frac{dC}{d\omega} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).e^{-j\omega t}.dt$$

and the corresponding **Symmetric Inverse Transform** is defined as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_s(\omega).e^{j\omega t}.d\omega$$

The Fourier Transform is useful in analysing transients in electrical circuits, especially where the elements are frequency dependant.

Fourier CosineTransform

A Fourier Cosine Transform $F_1(\omega)$ may be defined when the non-repetitive waveform is even.

f(t) = f(-t) so that

$$F_1(\omega) = \frac{2}{\pi}\int_0^{\infty} f(t).\cos\omega\, t \cdot dt \text{ , and}$$

$$f(t) = \int_0^{\infty} F_1(\omega).\cos\omega\, t \cdot d\omega$$

Fourier SineTransform

A Fourier Sine Transform $F_2(\omega)$ may be defined when the non-repetitive waveform is odd.

f(t) = –f(-t) so that

$$F_2(\omega) = \frac{2}{\pi}\int_0^{\infty} f(t).\sin\omega\, t \cdot dt \text{ , and}$$

$$f(t) = \int_0^{\infty} F_2(\omega).\sin\omega\, t \cdot d\omega$$

The Fourier Transform is sometimes expressed in terms of the sum of a sine and cosine series, instead of the exponential series.

$$f(t) = \int_0^{\infty} \left[A(\omega).\cos\omega t + B(\omega).\sin\omega\, t\right] \cdot d\omega$$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t).\cos\omega\, t \cdot dt$, and

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t).\sin\omega\, t \cdot dt$$

With functions which are non-repetitive, and does not decay till infinite time (such as the sine waveform or the cosine waveform), the Fourier Integral Transform may not be obtained.

To avoid this problem, waveforms which do not decay may be artificially decayed by an exponential factor to allow the integration. The integrated result is then exponentially magnified to correct for the initial decay introduced. However, such exponential magnification can also magnify numerical errors.

The Laplace Transform is defined based on this artificial decay.

Laplace Transform

In obtaining the Laplace Transform, any function f(t) is initially decayed artificially by an exponential factor $e^{-\sigma t}$, so that the new function always becomes integrable. However, the decay would correspond to an exponential rise (rather than a decay) with negative time. The Laplace transform is thus defined only for ***causal functions*** (functions that are caused and hence are of zero value before time zero).

The Laplace Transform of a time function f(t) is thus defined as

$$\mathcal{L}\,[f(t)] = F(s) = \int_0^{\infty} f(t) \cdot e^{-st} \cdot dt$$

where $s = \sigma + j\,\omega$ is the Laplace operator

The Laplace operator *s* is also considered as a complex frequency.

If we compare with the Fourier Transform pair with a multiplier of 2π, then the Laplace Inverse Transform takes the form

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\cdot e^{st}\cdot ds$$

It is seen that the form of the transform has simplified from that of the Fourier Transform. However, it is very rarely that the Inverse transform is calculated in this manner. It is generally obtained from a knowledge of the transforms of common functions, generally found in tabulated form.

The Laplace Transform is very useful in circuit transient analysis as it can convert differential equations to linear algebraic equations.

Response of a linear Passive Bilateral Network

Consider a linear passive bilateral two-port network to which an excitation e(t) is given at one port and which causes some response r(t) at the other port.

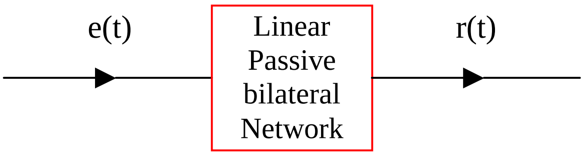


Figure 2 – Transfer Function

In general, the response r(t) would be related to the input e(t) by an ordinary linear differential equation.

$r(t) = F(p)\,.\,e(t)$ where $p = d/dt$ = differential operator

Consider an exponential excitation function e^{st}.

i.e. $e(t) = e^{st} \quad\rightarrow\quad r(t) = F(p)\,.\,e^{st} = e^{st}\,.\,F(s)$

Thus for an exponential excitation, the system has a transfer function r(t)/e(t) equal to F(s).

As stated earlier, any non-repetitive (or even repetitive) function may be broken up into a series of exponentials. The coefficients of these exponentials are given by the Laplace Transform.

Thus for any other excitation e(t), if the Laplace Transform E(s) is considered, it will be related to the Laplace Transform R(s) of the response r(t) by the transfer function F(s).

Thus for any causal excitation e(t),

$R(s) = F(s)\,.\,E(s)$

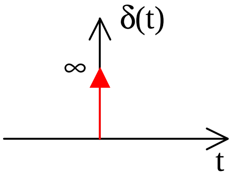
One of the advantages of the Laplace Transform is that it converts ordinary differential equations in to algebraic equations, so that the solution is fairly simple. The inverse transform is then obtained to get the time response.

Let us now consider the Laplace Transform of some special causal functions.

Laplace Transform of Special Causal Functions

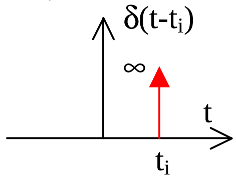
(a) *Unit impulse function δ(t)*

The unit impulse has a value 0 at all values of t other than at t = 0 where it has an infinite magnitude. Also the integral of the unit impulse function over time is equal to 1.



$$\mathcal{L}\,[\delta(t)] = \int_0^\infty \delta(t)\cdot e^{-st}\cdot dt = 1$$

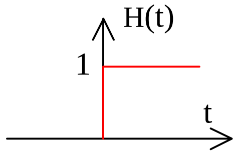
If the unit impulse occurs at t = t_i, rather than at t = 0, then the function is $\delta(t - t_i)$.



$$\mathcal{L}\,[\delta(t - t_i)] = \int_0^\infty \delta(t - t_i)\cdot e^{-st}\cdot dt = e^{-st_i}$$

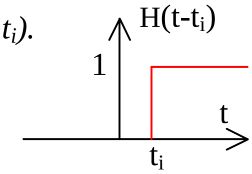
(b) *Unit step function H(t)*

The unit step has a value 0 for values of t < 0 and a value of 1 for t > 0.



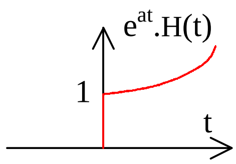
$$\mathcal{L}\,[H(t)] = \int_0^\infty H(t)\cdot e^{-st}\cdot dt = \int_0^\infty 1\cdot e^{-st}\cdot dt = \frac{1}{s}$$

If the unit step occurs at t = t_i, rather than at t = 0, then the function is $H(t - t_i)$.



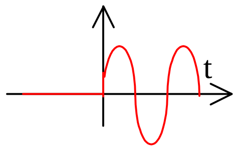
$$\mathcal{L}\,[H(t - t_i)] = \int_0^\infty H(t - t_i)\cdot e^{-st}\cdot dt = \int_{t_i}^\infty 1\cdot e^{-st}\cdot dt = \frac{e^{-st_i}}{s}$$

(c) *Causal exponential function* $e^{at}\,.\,H(t)$



$$\mathcal{L}\,[e^{at}.H(t)] = \int_0^\infty e^{at}.H(t)\cdot e^{-st}\cdot dt = \int_0^\infty 1\cdot e^{-(s-a)t}\cdot dt = \frac{1}{s-a}$$

(d) *Causal Sinusoidal function* $\sin(\omega t+\phi)\,.H(t)$



$$\mathcal{L}\,[\sin(\omega t+\phi).H(t)] = \int_0^\infty \sin(\omega t + \phi).H(t)\cdot e^{-st}\cdot dt$$

$$= \int_0^\infty \sin(\omega t + \phi).\cdot e^{-st}\cdot dt = F(s)$$

$$F(s) = \sin(\omega t + \phi).\frac{e^{-st}}{-s}\Bigg|_0^\infty - \int_0^\infty \omega\cdot\cos(\omega t + \phi)\cdot\frac{e^{-st}}{-s}\cdot dt$$

$$= \frac{\sin\phi}{s} + \frac{\omega}{s}\cdot\cos(\omega t + \phi)\cdot\frac{e^{-st}}{-s}\Bigg|_0^\infty - \int\frac{\omega^2}{s^2}\cdot\sin(\omega t + \phi)\cdot e^{-st}\cdot dt$$

$$= \frac{\sin\phi}{s} + \frac{\omega}{s^2}\cdot\cos\phi - \frac{\omega^2}{s^2}\cdot F(s)$$

$(s^2+\omega^2).\ F(s) = s\ .\ \sin\phi + \omega\ .\ \cos\phi$

$$F(s) = \frac{s.\sin\phi + \omega.\cos\phi}{s^2 + \omega^2}$$ with $\phi = 0^\circ$ and 90° the following are obtained.

$$\mathcal{L}\,[\sin\omega t.H(t)] = \frac{\omega}{s^2 + \omega^2}, \quad \mathcal{L}\,[\cos\omega t.H(t)] = \frac{s}{s^2 + \omega^2}$$

(e) *Laplace Transform of the causal derivative* $\frac{d\,f(t)}{dt}$

$$\mathcal{L}\ \frac{d\,f(t)}{d\,t} = \int_0^\infty \frac{d\,f(t)}{d\,t}\cdot e^{-st}\cdot dt = \int_0^\infty e^{-st}\cdot d\,f(t) = e^{-st}\cdot f(t)\Big|_0^\infty - \int_0^\infty f(t).(-s).e^{-st}.dt$$

$$= (-)\ f(0^-) + s.\int_0^\infty f(t).e^{-st}.dt$$

$$\mathcal{L}\left[\frac{d\,f(t)}{d\,t}\right] = s.F(s) - f(0^-)$$

It is worth noting that unlike in the case of the ordinary derivative, the transform of the derivative also keeps information about the initial condition [i.e. $f(0^-)$]

(f) *An exponential multiplication of* e^{at} *in the time domain*

An exponential multiplication of e^{at} in the time domain corresponds to a shift of ***a*** in the s-domain.

$$\mathcal{L}\ [e^{at}\ .\ f(t)] = \int_0^\infty e^{at}\cdot f(t).e^{-st}.dt\cdot f(t)\Big|_0^\infty = \int_0^\infty f(t).e^{-(s-a)t}.dt$$

$$= F(s-a)$$

(g) *A shift in the time domain*

A shift in ***a*** the time domain f(t – a).H(t – a), corresponds to an exponential decay in the s-domain.

$$\mathcal{L}\ [f(t-a).H(t-a)] = \int_0^\infty f(t-a).H(t-a).e^{-st}.dt$$

$$= e^{-sa}\int_{-a}^\infty f(t-a).H(t-a).e^{-s(t-a)}.d(t-a)$$

$$= e^{-sa}\int_{-a}^\infty f(\tau).e^{-s\tau}.d\tau$$

$$= e^{-as}\int_0^\infty f(\tau).e^{-s\tau}.d\tau \qquad \text{since } f(\tau) = 0 \text{ for } \tau < 0.$$

$$= e^{-as}.\ F(s)$$

(h) *For a periodic waveform f(t) with period T*

$$\mathcal{L}\ [f(t)] = \int_0^\infty f(t).e^{-st}.d\,t$$

$$= \int_0^T f(t).e^{-st}.d\,t + \int_T^{2T} f(t).e^{-st}.d\,t + \int_{2T}^{3T} f(t).e^{-st}.d\,t + \ldots\ldots$$

using a change of variables, this may be re-written as follows

$$\mathcal{L}\ [f(t)] = \int_0^T f(t).e^{-st}.d\,t + \int_0^T f(t+T).e^{-st}.d\,t + \int_0^T f(t+2T).e^{-st}.d\,t + \ldots\ldots$$

Since the function is periodic, $f(t) = f(t+T) = f(t+2T) = \ldots..$

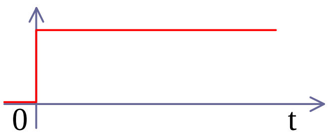
$$\mathcal{L}\,[f(t)] \quad = \int_0^T f(t).e^{-st}.dt + e^{-sT}\int_0^T f(t).e^{-st}.dt + e^{-2sT}\int_0^T f(t).e^{-st}.dt + \ldots\ldots$$

$$= [1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \ldots\ldots]\int_0^T f(t).e^{-st}.dt$$

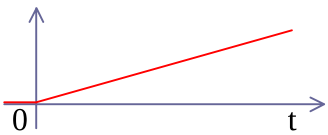
$$\mathcal{L}\,[f(t)] \quad = \frac{1}{1-e^{-sT}}\int_0^T f(t).e^{-st}.dt$$

The transforms of other ***causal functions*** may be similarly obtained, and the table gives the Laplace transforms for the common functions.

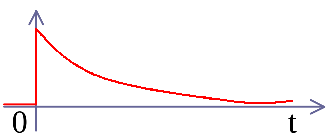
$\delta(t)$ unit impulse 1



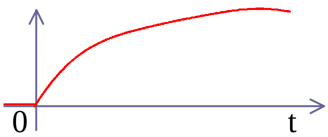
$H(t)$ unit step $\dfrac{1}{s}$



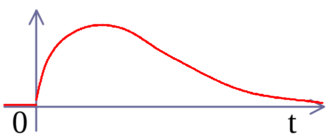
t ramp $\dfrac{1}{s^2}$



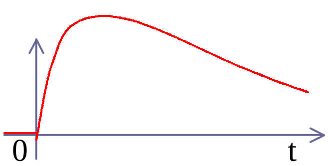
e^{-at} exponential decay $\dfrac{1}{s+a}$



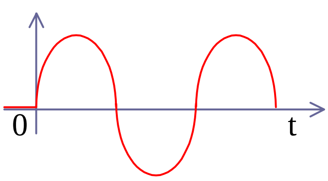
$1 - e^{-at}$ $\dfrac{a}{s(s+a)}$



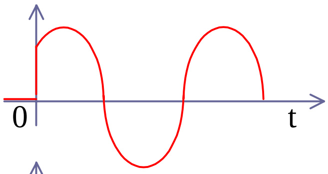
$t.e^{-at}$ $\dfrac{1}{(s+a)^2}$



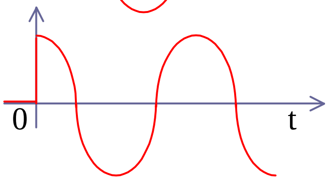
$e^{-at} - e^{-bt}$ double exponential $\dfrac{b-a}{(s+a)(s+b)}$



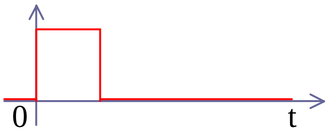
$\sin \omega t$ sine wave $\dfrac{s}{s^2+\omega^2}$



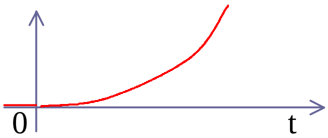
$\sin(\omega t + \phi)$ $\qquad$ $\dfrac{\omega\cos\phi + s\sin\phi}{s^2 + \omega^2}$



$\cos \omega t$ $\qquad$ cosine wave $\qquad$ $\dfrac{s}{s^2 + \omega^2}$



rectangular pulse $\qquad$ $\dfrac{1 - e^{-sT}}{s}$



t^n $\qquad$ n^{th} order ramp (n > 0) $\qquad$ $\dfrac{n!}{s^{n+1}}$

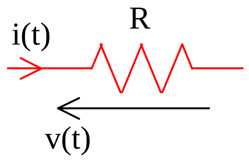
sinh at	hyperbolic sine	$\dfrac{a}{s^2 - a^2}$
cosh at	hyperbolic cosine	$\dfrac{s}{s^2 - a^2}$
$a.f_1(t) + b.f_2(t)$	addition	$a.F_1(s) + b.F_2(s)$
$\dfrac{d\ f(t)}{dt}$	first derivative	$s\,F(s) - f(0^-)$
$\dfrac{d^n\ f(t)}{dt^n}$	n^{th} derivative	$s^n\,F(s) - \sum_{j=1}^{n} s^{n-j}\dfrac{d^{j-1}f}{dt^n}(0^-)$
$\int_{0^-}^{t} f(t)\cdot dt$	definite integral	$\dfrac{1}{s}F(s)$
$t.f(t)$		$-\dfrac{d\,F(s)}{d\,s}$
$(-t)^n.f(t)$		$\dfrac{d^n\ F(s)}{d\,s^n}$
$e^{-\alpha t}.f(t)$	exponential multiplier	$F(s+\alpha)$
$f(t-\tau)$	shift	$e^{-s\tau}.F(s)$
	periodic function (period T)	$\dfrac{1}{(1-e^{-sT})}\int_{0^-}^{T} f(t).e^{-st}.dt$

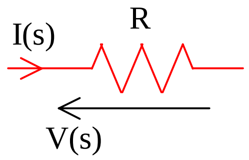
Transient Analysis of Circuits using the Laplace Transform

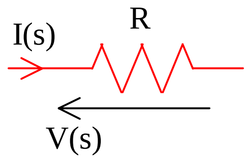
Electrical Circuits are usually governed by linear differential equations. Since derivatives and integrals get converted to multiplications and divisions in the s-domain, the solution of circuit equations can be converted to the solution of algebraic equations.

Let us first consider the representation of the three basic circuit components in Laplace Transform analysis.

(a) ***Resistive Element R***







$v(t) = R \, . \, i(t)$

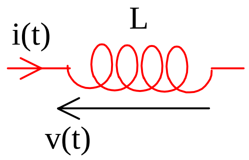
$$i(t) = \frac{1}{R} v(t)$$

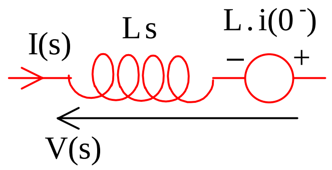
$V(s) = R \, . \, I(s)$

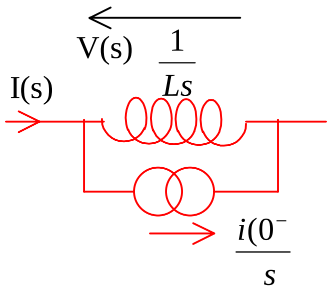
$$I(s) = \frac{1}{R} V(s)$$

Thus the resistor may be represented by an impedance of value R even in the s-domain.

(b) ***Inductive Element L***







$$v(t) = L \, . \frac{d\, i(t)}{d\, t}$$

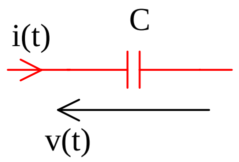
$V(s) = L\, s. \, I(s) - L.i(0^-)$

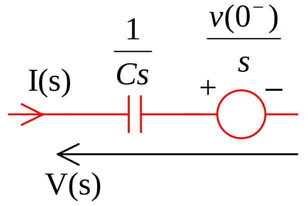
$$i(t) = \frac{1}{L}\int v(t).dt + i(0^-)$$

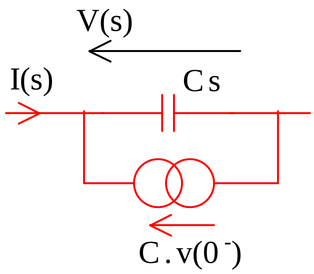
$$I(s) = \frac{1}{Ls}.V(s) + \frac{i(0^-)}{s}$$

Thus the inductor may be represented by an impedance of value $L\ s$ and either a series voltage source or a parallel current source. These sources represent the initial energy stored in the inductor at time t = 0. It is to be noted that the initial current $i(0^-)$ appears in both forms of the equation and that one form can be obtained algebraically from the other, without resorting to any additional information.

(c) ***Capacitive Element L***







$$v(t) = \frac{1}{C}\int i(t).dt + v(0^-)$$

$$V(s) = \frac{1}{Cs}.I(s) + \frac{v(0^-)}{s}$$

$$i(t) = C \, . \frac{d\, v(t)}{d\, t}$$

$I(s) = C\, s. \, I(s) - C.v(0^-)$

Thus the capacitor may be represented by an impedance of value $\frac{1}{Cs}$ and either a series voltage source or a parallel current source. As with the inductor these sources represent the initial energy stored in the capacitor at time t = 0.

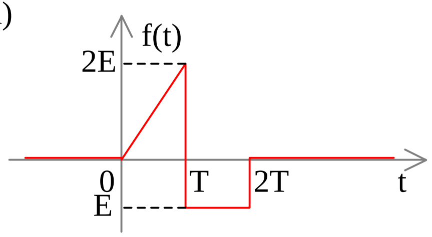
In addition to Ohm's Law and Kirchoff's laws, Superposition, Thevenin's and Norton's theorems may also be applied to these transformed circuits in the s-domain.

Using these circuits, and the transforms of source voltages and/or currents, the system transients could be obtained. You would by now have realised that this method is much less tedious than the solution of the differential equations to find the transient solutions and then substituting the initial and final conditions applicable.

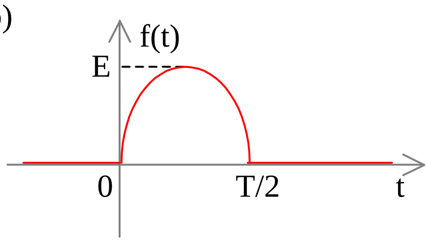
Example 1

Find the Laplace transform of the following waveforms.

(a)



(b)



f(t) = E sin ωt for 0 < t < T/2
f(t) = 0 elsewhere

Solution

(a)

using first principles

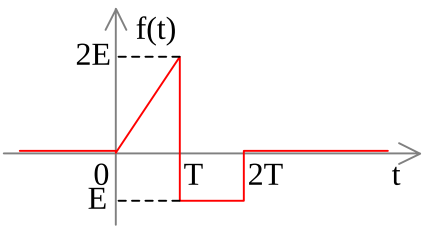
$$\mathcal{L}[f(t)] = \int_0^\infty f(t).e^{-st}.dt = \int_0^T 2E.\frac{t}{T}.e^{-st}.dt + \int_T^{2T}(-E).e^{-st}.dt + 0$$

$$= 2E.\frac{t}{T}.\frac{e^{-st}}{-s}\bigg|_0^T - \int_0^T 2E.\frac{1}{T}.\frac{e^{-st}}{-s}.dt - E.\frac{e^{-st}}{-s}\bigg|_T^{2T} = \frac{2E.e^{-sT}}{-s} - \frac{2E}{T}.\frac{e^{-st}}{(-s)^2}\bigg|_0^T + \frac{E}{s}.(e^{-2sT} - e^{-sT})$$

$$= \frac{2E.e^{-sT}}{-s} - \frac{2E}{T}.\frac{(e^{-sT}-1)}{(-s)^2} + \frac{E}{s}.(e^{-2sT} - e^{-sT})$$

$$= \frac{E}{s}\left[e^{-2sT} - 3e^{-sT}\right] + \frac{2E}{s^2T}.\left[1 - e^{-sT}\right]$$

using properties from tables (this is not always possible)



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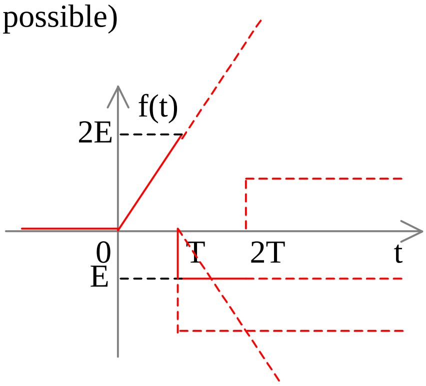


The part of the ramp from t = 0 to T can be considered as the addition of a positive ramp at t=0, a negative ramp at t = T and a negative step of magnitude 2E at time t = T. The remaining part of the waveform can be considered to be made up of a negative step waveform of magnitude E at t = T, and a positive step also of magnitude E at t = 2T. Superposition of these waveforms will give the resultant waveform.

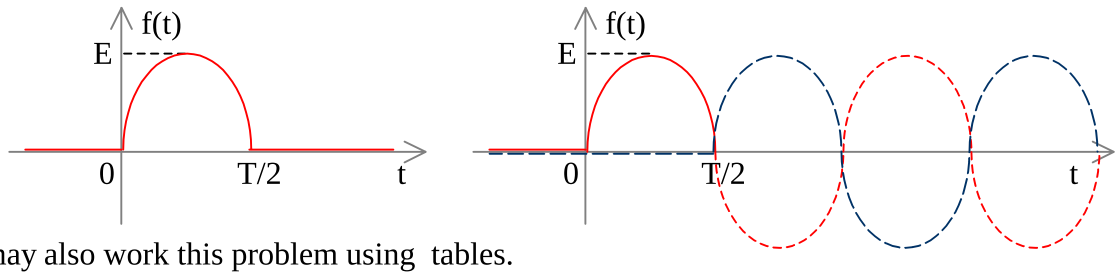
These will have Laplace transforms which will add up as follows.

$$\mathcal{L}[f(t)] = \left(\frac{2E}{T}.\frac{1}{s^2} - \frac{2E}{T}.\frac{1}{s^2}.e^{-sT} - \frac{2E}{s}.e^{-sT}\right) - \left(\frac{E}{s}.e^{-sT} + \frac{E}{s}.e^{-2sT}\right)$$

$$= \frac{E}{s}\left[e^{-2sT} - 3e^{-sT}\right] + \frac{2E}{s^2T}.\left[1 - e^{-sT}\right]$$

which is the identical result that was obtained from the normal method.

(b)

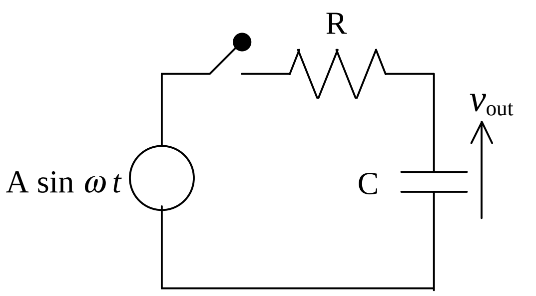


We may also work this problem using tables.

The given waveform can also be considered as been built up of a causal sine wave starting at t = 0, and a negative of that waveform starting at T/2.

Thus the transform of the waveform is given by

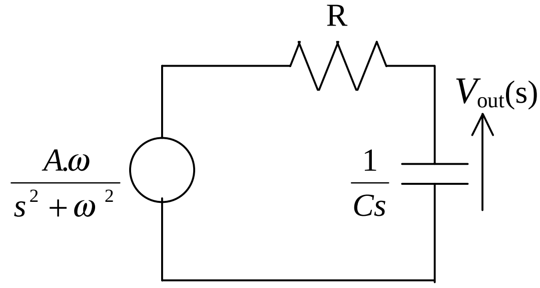
$$\mathcal{L}[f(t)] = \frac{s}{s^2+\omega^2} - \frac{s}{s^2+\omega^2}.e^{-sT/2} = \frac{s}{s^2+\omega^2}\left[1 - e^{-sT/2}\right]$$



Example 2

Determine the transient voltage appearing across the capacitor when the switch is closed at time t = 0. Capacitor C is initially uncharged.

Solution



The transformed circuit is shown. The capacitor has not been associated with a source as there is no initial charge (or voltage) on the capacitor.

Using potential divider action

$$\frac{V_{out}(s)}{A\omega/(s^2+\omega^2)} = \frac{1/Cs}{R + 1/Cs} = \frac{1}{1+RCs}$$

$$\therefore V_{out}(s) = \frac{A}{1+RCs}.\frac{\omega}{s^2+\omega^2} = \frac{A\alpha}{\alpha+s}.\frac{\omega}{s^2+\omega^2}, \quad \text{where } \alpha = \frac{1}{RC}$$

This can be split up as follows.

$$V_{out}(s) = \frac{A.\omega.\alpha}{\omega^2+\alpha^2}\left[\frac{1}{s+\alpha} - \frac{s-\alpha}{s^2+\omega^2}\right]$$

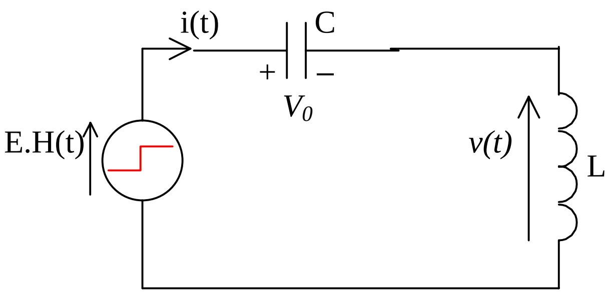
Using the tables, the inverse transform is then given as

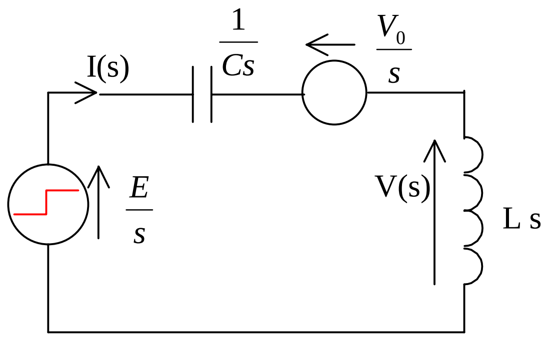
$$v_{out}(s) = \frac{A.\omega.\alpha}{\omega^2+\alpha^2}\left[e^{-\alpha t} - \cos\omega t + \frac{\alpha}{\omega}\sin\omega t\right]$$

Example 3

In a series LC circuit, initially the capacitor is charged to a voltage of V_o and the inductor does not carry any current. At time t = 0, a step voltage of magnitude E is applied to the series combination. Determine the transient voltage across L.

Solution





The circuit is first transformed to the Laplace domain. The voltage source form is selected for the capacitance because the circuit is a series circuit and that form makes calculations easier. Since there is no initial current in the inductor, no source has been associated with the inductor.

$$I(s) = \frac{\frac{E}{s} - \frac{V_0}{s}}{Ls + \frac{1}{Cs}}$$

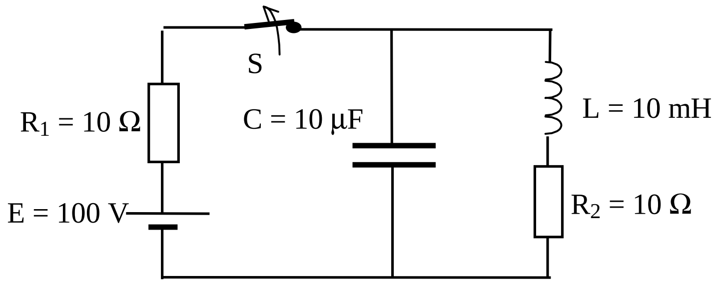
$$\text{V(s)} = \text{L s.I(s)} = Ls \cdot \frac{\frac{E}{s} - \frac{V_0}{s}}{Ls + \frac{1}{Cs}} = LCs \cdot \frac{E - V_0}{LCs^2 + 1} = (E - V_0) \cdot \frac{s}{s^2 + \frac{1}{LC}}$$

Let $\frac{1}{LC} = \omega_0^2, \quad \therefore V(s) = (E - V_0) \cdot \frac{s}{s^2 + \omega_o^2}$

$$\therefore v(t) = (E - V_0) \cdot \cos\sqrt{\frac{1}{LC}}\,t$$

Example 4

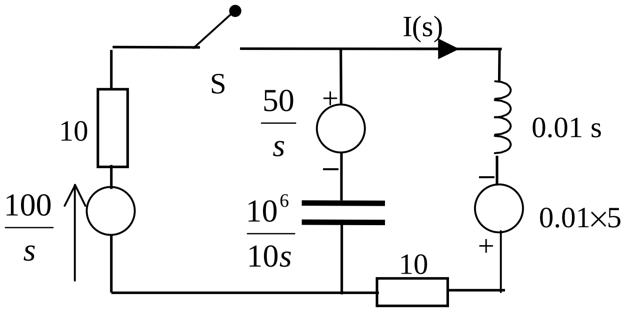
Figure shows a circuit which has reached steady state with switch closed. If the switch S is opened at time t=0, obtain an expression for the ensuing current through the inductor.



Solution

From potential divider action, under steady state conditions, half the supply voltage will drop across R_1 and half the voltage across R_2. Therefore the voltage across the capacitor initially will be 100/2 = 50 V, and the inductor current will be 100/20 = 5 A.

Transform the circuit to the Laplace domain.



Note the directions of the two sources. These correspond to the directions of the initial voltage across the capacitor and the initial current through the inductor.

Note also that since switch S is now opened at t = 0, only the other two branches will become part of the circuit.

Thus $\quad \text{I(s)} = \dfrac{\dfrac{50}{s}+0.05}{0.01s+10+\dfrac{10^5}{s}} = \dfrac{50+0.05s}{0.01s^2+10s+10^5} = \dfrac{5000+5s}{s^2+1000s+10^7}$

$$\text{I(s)} = \frac{5000+5s}{(s+500)^2+3122.5^2} = \frac{5(s+500)}{(s+500)^2+3122.5^2} + \frac{0.8006\times 3122.5}{(s+500)^2+3122.5^2}$$

Finding the inverse transform from the standard expressions,

$$\text{i(t)} = 5e^{-500t}\cos 3122.5t + 0.8006e^{-500t}\sin 3122.5t \quad \text{A}$$