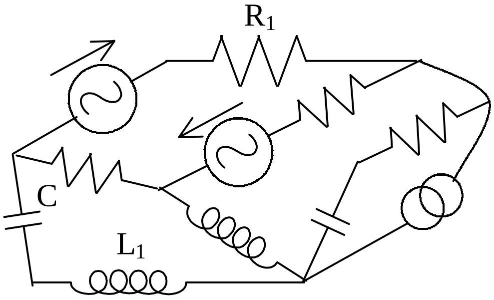
Matrix Analysis of Networks

It is tedious to analyse large network using normal equations. It is easier and more convenient to formulate large networks in matrix form.

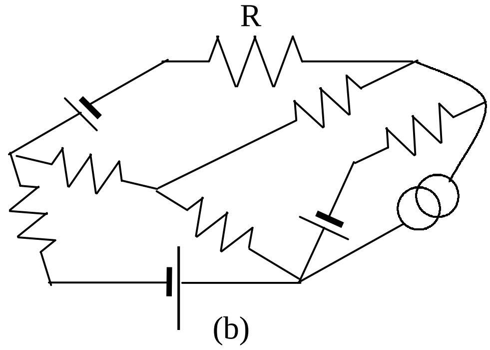
To have a neat form of solution, it is necessary to know the structure of a network and formulate the problem based on the structure. Topology deals with the structure of an interconnected system, and formulates the problem based on non-measurable properties of the network.

The geometric structure of the interconnection of the network elements completely characterises the number of independent loop currents or the number of independent node-pair voltages that are necessary to study the network.

Consider the two networks shown in figure 1(a) and (b).



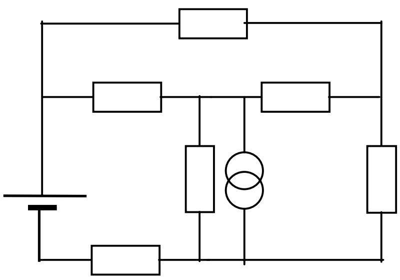
(a)



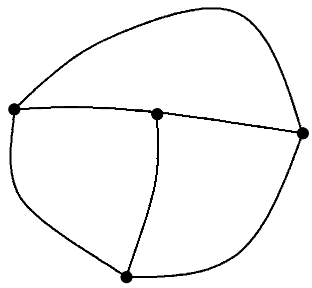
(b)

Figure 1 – Structure of the network

If you look carefully at the structure of the two circuits, you will notice that they both have the same structure (or topology). However the elements are quite different. In fact even the circuit shown in figure 2(a) has the same topology.



(a) Network

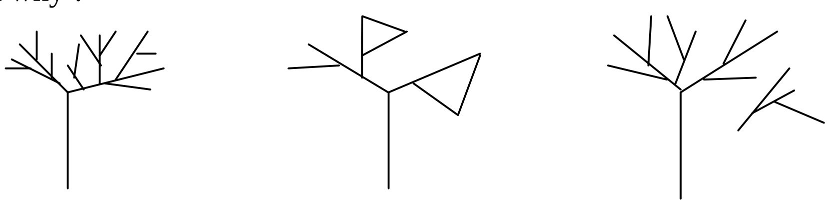


(b) Graph of Network

Figure 2 – Circuit

Figure 2(b) shows the structure corresponding to each of the 3 circuits, but without indicating any of the elements in the network. Such a diagram is known as the graph of the network. It has all the nodes of the original network. In obtaining the graph, each element of the network is represented by a line, each voltage source by a short-circuit and each current source by an open circuit.

Which of the following diagrams would say could represent a normal tree (without leaves) ? and why ?

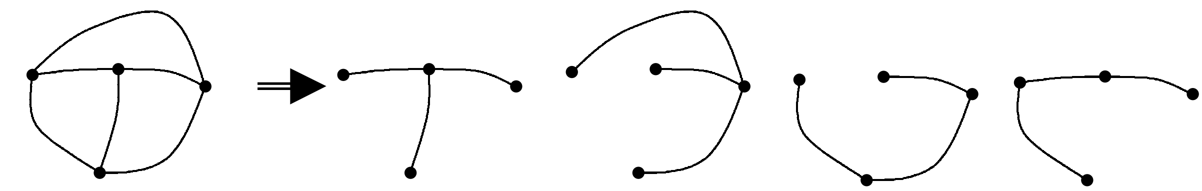


Only the first diagram would fully satisfy the requirements of a normal tree. The second diagram has branches closing on itself (only a tree like Nuga might appear to close on itself). The third has branches which are in mid air not joined to the main tree. Thus we see that there are certain properties that we associate with trees. These are

1. All branches must be part of the tree
2. There cannot be closed loops formed from branches
3. There cannot be branches which are isolated from the tree

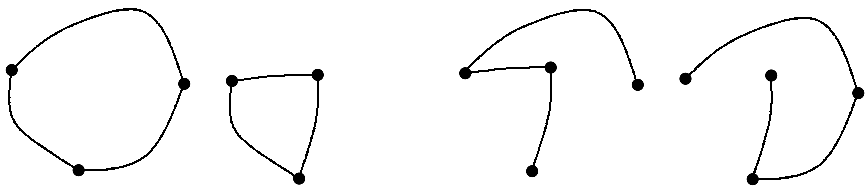
We apply these same properties when we define a tree of a given network. However, unlike the graph, there can be many trees associated with a given network. Also unlike in the case of a natural tree, the tree of a network need not have a trunk coming from the ground and branches coming from the trunk.

A tree of a network is a graph of the network with some of the links removed in such a way so as to leave all the nodes connected together by the graph, but so as not to have any loop left in the network. For each network graph, there are a number of possible trees. Some of the trees are shown below.



Graph of Network Some of the possible trees

When a tree of the network is removed from the graph, what remains is called the co-tree of the network. It is the graph of the removed links and is the compliment of the tree. Unlike a tree of a network, a co-tree may contain closed loops, and disconnected branches. The corresponding co-trees of the above trees are shown below.



Let b = number of branches in the network

n = number of nodes in the network

l = number of independent loops

A single branch is required to join two nodes. Joining each additional node would require an additional branch. Thus the number of branches in a tree would be one less than the number of nodes.

number of branches in tree = $n - 1$

number of links removed = $b - (n - 1) = b - n + 1$

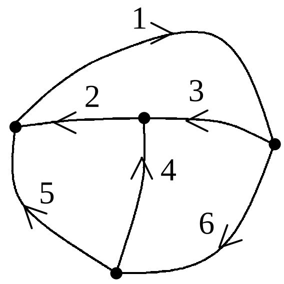
If we add any one of the removed links to a tree, then a loop is formed. Since this involves the link that was added this is part of an independent loop as it could not have formed part of any of the loops that were already there.

Therefore the number of links removed from the graph to form the tree is equal to the number of independent loops.

i.e. $l = b - n + 1$

In analysing network, if the branches are not numbered and they are not assigned directions, it would not be possible to formulate the equations governing the circuit. The numbering and the directions assigned to a graph is not unique, unlike the graph itself.

An oriented graph for the earlier problem is shown in the figure. The direction marked is that of the current, and the voltage is considered to increase in the direction opposite to the flow of current and is marked as such in the oriented graph.

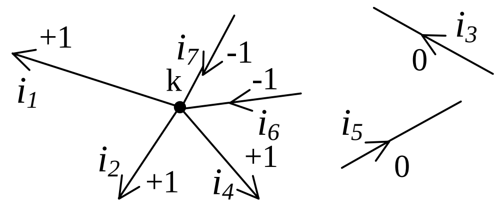


Oriented Graph

Matrix Analysis of Networks

When we solve circuit problems, we need to write the equations corresponding to the Ohm's Law and the Kirchoff's Laws. These same equations need to be used even when there are a large number of branches, and as such when using Matrix analysis of networks.

Kirchoff's current Law in matrix form



Let I_j be the current in the j^{th} branch.

Consider any node k in the network, as shown in the figure. The j^{th} branch may either be directed away from k^{th} node, may be directed towards k^{th} node, or may not be incident on the k^{th} node.

From Kirchoff's current Law,

$$-i_1 - i_2 - i_4 + i_6 + i_7 = 0 \quad \text{or} \quad i_1 + i_2 + i_4 = i_6 + i_7 \quad \text{or}$$

$$i_1 + i_2 + i_4 - i_6 - i_7 = 0 \quad \text{or} \quad +1 . i_1 + 1 . i_2 + 0 . i_3 + 1 . i_4 + 0 . i_5 - 1 . i_6 - 1 . i_7 = 0$$

[These are different forms of the same equation].

The last form is preferred for matrix implementation, as all the currents in the network are represented in the equation with different coefficients. The equation could also have been written with all the coefficients negated. However, if it is to be used for computer implementation, there must be a unique method of obtaining the coefficients. The convention used is as follows.

Branch currents directed away from the node are associated with a coefficient of +1, while current directed towards the node are associated with a coefficient –1. Branches not connected to (or incident on) the node are obviously associated with a coefficient of 0.

These coefficients are defined as a_{jk} so that Kirchoff's current law may thus be written as

$$a_{1k} . i_1 + a_{2k} . i_2 + a_{3k} . i_3 + a_{4k} . i_4 + \ldots\ldots\ \ldots\ldots\ \ldots\ a_{7k} . i_7 = 0 \qquad \text{for the } k^{th} \text{ node}$$

or $$\sum_{j=1}^{b} a_{jk} \cdot i_j = 0 \qquad \text{at } k^{th} \text{ node, for all k; where } a_{jk} = -1, 0, \text{ or } +1$$

The collection of such equations for each node k, would give the matrix equation

$$\underset{(n\times b)}{[A]^t} \cdot \underset{(b\times 1)}{\underline{I_b}} = \underset{(b\times 1)}{\underline{0}}$$

In the matrix $[A]^t$, the row vectors are dependant, since their sum is zero. For this reason, the matrix $[A]^t$ is written with one row less, usually the last row for convenience, so that there is only (n-1) rows.

The matrix $[A]^t$ of dimension (n-1)×b is referred to as the node-branch incidence matrix.

The matrix [A] of dimension b× (n-1) is then referred to as the branch-node incidence matrix.

The elements of the matrix [A] are

$a_{jk} = +1$ if j^{th} current is **directed away** from the k^{th} node

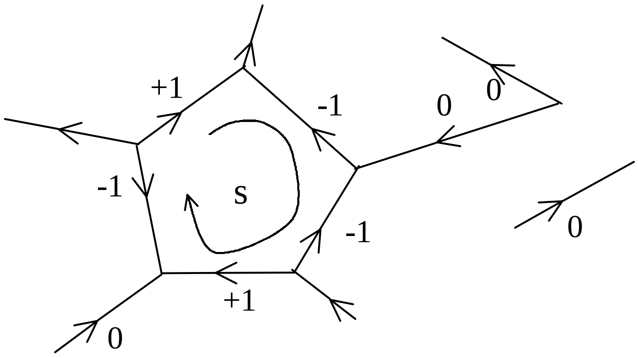
$a_{jk} = -1$ if j^{th} current is **directed towards** the k^{th} node

$a_{jk} = 0$ if j^{th} current is **not incident** on the k^{th} node

Kirchoff's voltage Law in matrix form

Similarly, the Kirchoff's voltage law may be applied for the s^{th} loop.

The coefficients b_{rs} of the voltages are defined as +1, −1 or 0 depending on whether the loop direction is the same as the branch direction or not.



Thus $\sum_{r=1}^{b} b_{rs} \cdot v_r = 0$ for s^{th} node, for all s;

where $b_{rs} = -1, 0,$ or $+1$

In matrix form this becomes

$$\underset{(l\times b)}{[B]^t} \cdot \underset{(b\times 1)}{\underline{V_b}} = \underset{(l\times 1)}{\underline{0}}$$

The matrix $[B]^t$ of dimension $l\times b$ is referred to as the mesh-branch incidence matrix.

The matrix [B] of dimension $b\times l$ is then referred to as the branch-mesh incidence matrix.

The elements of the matrix [B] are

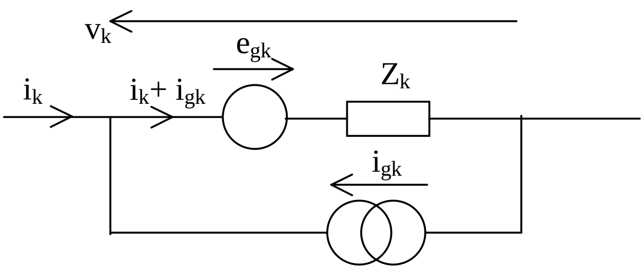
$b_{rs} = +1$ if r^{th} current is in the **same direction** as the s^{th} loop

$b_{rs} = -1$ if r^{th} current is in the **opposite direction** to the s^{th} loop

$b_{rs} = 0$ if r^{th} current is in the **not part of** the s^{th} loop

Ohm's Law in matrix form

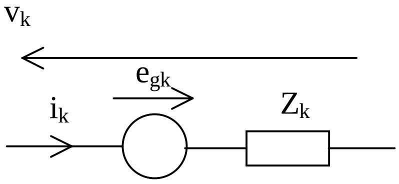
A general branch may contain a voltage source and/or a current source in addition to the branch impedance/admittance.



Applying Ohm's Law

$$v_k = -\,e_{gk} + Z_k\, i_{gk} + Z_k\, i_k \qquad \text{for all branches } k = 1, 2, \ldots\ldots b$$

However, using Thevenin's Theorem or Norton's Theorem, a current generator may be converted to a voltage generator or vice versa, except when ideal. Thus analysis can be done with only either a voltage source or a current source.

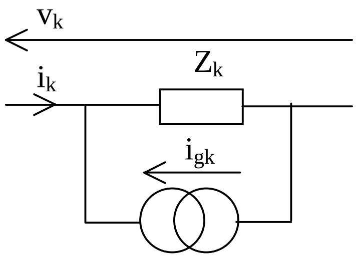


Thus we may either write

$$v_k = -\,e_{gk} + Z_k\, i_k \qquad \text{for all branches } k = 1, 2, \ldots\ldots b$$

which may be written in matrix form as

$$\underline{V_b} = -\underline{E_{gb}} + [Z_b]\underline{I_b}$$



or we may write

$$v_k = Z_k\, i_{gk} + Z_k\, i_k \qquad \text{or} \qquad Y_k\, v_k = Y_k Z_k\, i_{gk} + Y_k Z_k\, i_k$$

i.e. $\quad i_k = Y_k\, v_k - i_{gk} \qquad$ for all branches $k = 1, 2, \ldots\ldots b$

or in matrix form as

$$\underline{I_b} = -\underline{I_{gb}} + [Y_b]\underline{V_b} \;, \qquad \text{where } [Y_b] = [Z_b]^{-1}$$

In Summary

From Kirchoff's Laws we have

$$\underset{(n\times b)}{[A]^t} \cdot \underset{(b\times 1)}{\underline{I_b}} = \underset{(b\times 1)}{\underline{0}} \qquad (1) \qquad (n\text{-}1) \quad \text{independent equations}$$

$$\underset{(l\times b)}{[B]^t} \cdot \underset{(b\times 1)}{\underline{V_b}} = \underset{(l\times 1)}{\underline{0}} \qquad (2) \qquad l \quad \text{independent equations}$$

and from Ohm's Law

$$\underline{V_b} = -\underline{E_{gb}} + [Z_b]\underline{I_b} \qquad (3) \qquad b \quad \text{independent equations}$$

or $\quad \underline{I_b} = -\underline{I_{gb}} + [Y_b]\underline{V_b} \qquad (3)^* \qquad b \quad$ independent equations

Thus the total number of independent equations is $\quad n - 1 + l + b = b + b = 2\,b$

It is seen that there is a total of *2b* independent equations and *2b* unknowns (corresponding to *b* branch currents and *b* branch voltages). However, it is not usual to solve the equations for both current and voltage simultaneously. So let us see how these may be reduced for solution. The reductions can be done in one of two ways. The first is to eliminate the voltages and solve for currents. The second method is to eliminate the currents and solve for the voltages. These two methods are known as mesh analysis and nodal analysis respectively.

Mesh Analysis

In mesh analysis, we eliminate the branch voltages from the equations. We further reduce the number of remaining currents to the minimum using Kirchoff's current law and apply Kirchoff's voltage law for solution.

Let us define a set of mesh currents, $\underline{I_m}$.

The branch currents $\underline{I_b}$ can be seen to be related to the mesh currents $\underline{I_m}$ by an algebraic summation. In fact, if an example is considered, it will be easily seen that the relation corresponds to the matrix [B].

$$\underline{I_b} = [B]\underline{I_m} \qquad (4)$$

We can eliminate $\underline{V}_b$ from the equations, by pre-multiplying equation (3) by $[B]^t$.

i.e. $[B]^t \underline{V_b} = -[B]^t \underline{E_{gb}} + [B]^t [Z_b]\underline{I_b}$

from equation (2), it can be seen that $[B]^t V_b = 0$. Also $\underline{I_b} = [B]\underline{I_m}$

$\therefore \quad [B]^t \underline{E_{gb}} = [B]^t [Z_b][B]\underline{I_m}$

$[B]^t V_b = 0$ corresponds to the sum of the voltages around a loop is zero.

$\therefore \quad [B]^t V_b$ must correspond to the sum of the voltages around a loop.

Since $[B]^t$ is constant, and $\underline{E}_{gb}$ is the branch source voltage vector, $[B]^t E_{gb}$ must correspond to the sum of the source voltages around a loop. This is defined as the mesh source voltage vector $\underline{E}_{gm}$.

i.e. $\underline{E}_{gm} = [B]^t E_{gb}$

$\therefore \quad \underline{E_{gm}} = [B]^t [Z_b][B]\underline{I_m} = [Z_m]\underline{I_m}$

where $[Z_m] = [B]^t [Z_b][B]$

This corresponds to only *l* equations, and the only unknowns in that equation is $\underline{I_m}$ which corresponds to *l* unknowns.

Thus the original problem of 2b equations and 2b unknowns has been reduced to that of *l* equations and *l* unknowns.

The elements of the matrix $[Z_m]$ can be obtained either by going through the above mathematical considerations, or by inspection in the following manner for the simple problems.

z_{jj} = self impedance of mesh j

= sum of all branch impedances in mesh j

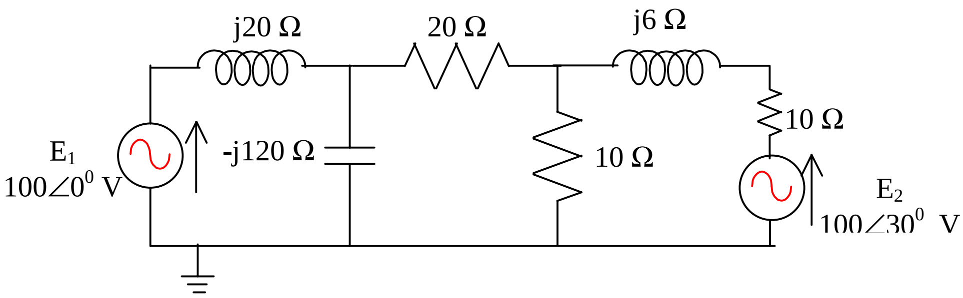
z_{jk} = mutual impedance between mesh j and mesh k

= sum of all branch impedances common to mesh j and mesh k and traversed in mesh direction − sum of all branch impedances common to mesh j and mesh k and traversed in opposite direction

e_j = algebraic sum of the branch voltage sources in mesh j in mesh direction.

Note: Matrix [B] is also known as the tie-set matrix (as its elements **tie** the loop together).

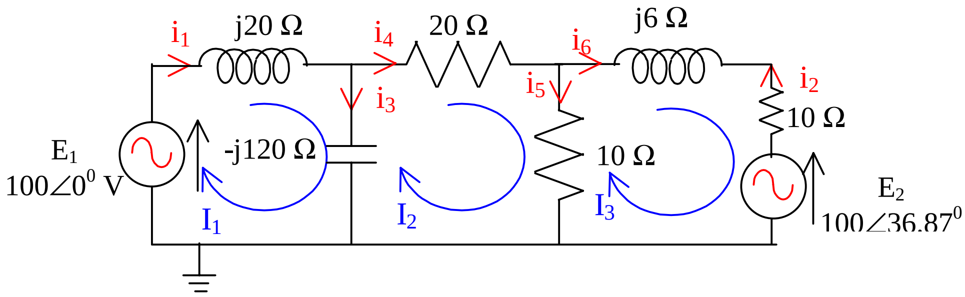
Example 1



Solve the circuit using Mesh matrix analysis. Work from first principles.

Solution

Let us first number the branches and the loops.



You will notice that I have used capital letters for the loop currents and simple letters for the branch currents. This is for convenience in not having to write a suffix m for mesh and b for branch.

Let us write the loop currents in terms of the branch currents.

$$
\begin{aligned}
i_1 &= I_1 \\
i_2 &= -I_3 \\
i_3 &= I_1 - I_2 \\
i_4 &= I_2 \\
i_5 &= I_2 - I_3 \\
i_6 &= I_3
\end{aligned}
\qquad \text{or in matrix form} \qquad
\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}
$$

This gives us the [B] matrix or the **Branch-Mesh incidence matrix**.

We can also write the **Mesh–Branch incidence matrix** $[B]^t$ matrix as follows, independent of the above, by writing the relation between the mesh direction and the branch direction.

$$
[B]^t = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}
$$

You will notice that this corresponds to the transpose of the matrix written earlier.

The vector of branch currents $\underline{E}_{gb}$ can be written as follows.

$$
\begin{bmatrix} 100\angle 0^0 \\ 100\angle 36.87^0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

We now need to write the branch impedance matrix.

Then we can write expressions for the mesh voltage vector and the mesh impedance matrix using the derived equations.

$$
\underline{E}_{gm} = [B]^t \, \underline{E}_{gb}, \qquad \text{and} \qquad [Z_m] = [B]^t \, [Z_b] \, [B]
$$

The Branch impedance matrix is

$$[Z_b]=\begin{bmatrix} j20 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & -j120 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & j6 \end{bmatrix}$$

Thus we can write the mesh source voltage vector

$$\underline{E_{gm}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 100\angle 0^0 \\ 100\angle 36.87^0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \underline{E_{gm}} = \begin{bmatrix} 100\angle 0^0 \\ 0 \\ -100\angle 36.87^0 \end{bmatrix}$$

This could also have been written by inspection by writing down the sources effectively driving the mesh currents round the loops. If first principles is not asked for, writing by inspection would generally be sufficient.

The mesh impedance matrix can be written as

$$[Z_m]=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} j20 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & -j120 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & j6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. $$[Z_m]=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} j20 & 0 & 0 \\ 0 & 0 & -10 \\ -j120 & j120 & 0 \\ 0 & 20 & 0 \\ 0 & 10 & -10 \\ 0 & 0 & j6 \end{bmatrix} = \begin{bmatrix} -j100 & j120 & 0 \\ j120 & 30-j120 & -10 \\ 0 & -10 & 20+j6 \end{bmatrix}$$

This matrix too could have been written by inspection, and would normally have been done that way, if first principles were not required.

Thus we may now write the mesh analysis equation as

$$\begin{bmatrix} 100\angle 0^0 \\ 0 \\ -100\angle 36.87^0 \end{bmatrix} = \begin{bmatrix} -j100 & j120 & 0 \\ j120 & 30-j120 & -10 \\ 0 & -10 & 20+j6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The equations may be solved by inversion or otherwise.

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} (30-j120)(20+j6)-10^2 & -j120(20+j6) & -j120\times 10 \\ -j120(20+j6) & -j100(20+j6) & -j100\times 10 \\ -j120\times 10 & -j100\times 10 & -j100(30-j120)-(j120)^2 \end{bmatrix}\begin{bmatrix} 100\angle 0^0 \\ 0 \\ -100\angle 36.87^0 \end{bmatrix}$$

i.e. $$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} 1220-j2220 & -j2400+j720 & -j1200 \\ -j2400+720 & -j2000+600 & -j1000 \\ -j1200 & -j1000 & -j3000-12000+14400 \end{bmatrix}\begin{bmatrix} 100 \\ 0 \\ -80-j60 \end{bmatrix}$$

The discriminant Δ is given by

Δ = (1220 – j2220)×(-j100) + (720 – j2400) ×(j120) + (-j1200) ×0

Δ = – j122000 – 222000 + j 86400 + 288000 = 66000 – j 35600 = 74989∠-28.34°

Thus I_1 is given as

I_1 = (122000 – j 222000 + 0 + j 96000 – 72000)/74989∠-28.34°

= (50000 – j 126000)/ 74989∠-28.34° = 135558∠-68.36° /74989∠-28.34°

= 1.808 ∠-40.02° A

[*Note:* The inversion has not been double checked]

Currents I_2 and I_3 can be similarly determined.

The branch currents i_1, i_2, ... may then be determined from the matrix equation. [Normally branch 6 would not have been marked as a separate branch, but as part of branch 2]

Nodal Analysis

In nodal analysis, we eliminate the branch currents from the equations. We further reduce the number of remaining voltages to the minimum using Kirchoff's voltage law and apply Kirchoff's current law for solution.

Let us define a set of nodal voltages, $\underline{V_N}$ which are node pair voltages (i.e. voltage across a pair of nodes)

The branch voltages $\underline{V_b}$ can be seen to be related to the nodal voltages $\underline{V_N}$ by an algebraic summation. In fact, if an example is considered, it will be easily seen that the relation corresponds to the matrix [A].

$$\underline{V_b} = [A]\underline{V_N} \qquad (5)$$

As was mentioned earlier, [A] too does not have the reference node.

Let us pre-multiply equation (3)* by $[A]^t$. This gives

$$[A]^t \underline{I_b} = -[A]^t \underline{I_{gb}} + [A]^t [Y_b]\underline{V_b}$$

from equation (1), $[A]^t \underline{I_b} = \underline{0}$. Also substituting from (5) we have

$$0 = -[A]^t \underline{I_{gb}} + [A]^t [Y_b][A]\underline{V_N}$$

i.e. $[A]^t \underline{I_{gb}} = [A]^t [Y_b][A]\underline{V_N}$ or $\underline{I_{gN}} = [Y_N]\underline{V_N}$

where $\underline{I_{gN}} = [A]^t \underline{I_{gb}}$, and $[Y_N] = [A]^t [Y_b][A]$

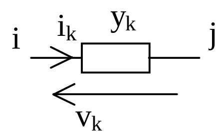
As in the case of mesh analysis, the source nodal current vector $\underline{I_{gN}}$ and the nodal admittance matrix $[Y_N]$ could be written by inspection as follows.

y_{ii} = sum of all branch admittances incident at node i

y_{ij} = negative of the sum of all branch admittances connecting node i and nodej .

The reason why the negative sign appears can be understood as follows.

Consider the simple circuit shown.



$i_k = y_k\, v_k = y_k\,(V_i - V_j)$

At any node i, from Kirchoff's current law,

nodal injected current at the node i is equal to the sum of the branch currents going out from the node

$I_{gi} = \Sigma\, i_k$ which gives the required expression for the nodal current vector

Also,

$I_{gi} = \Sigma\, i_k = \Sigma\, y_k\,(V_i - V_j)$

i.e. $I_{gi} = \sum_{\substack{j=1\\ j\neq i}}^{N} y_k V_i - \sum_{\substack{j=1\\ j\neq i}}^{N} y_k V_j$ at node i for all j (different k actually correspond to different j)

Since V_i is a constant for a given i,

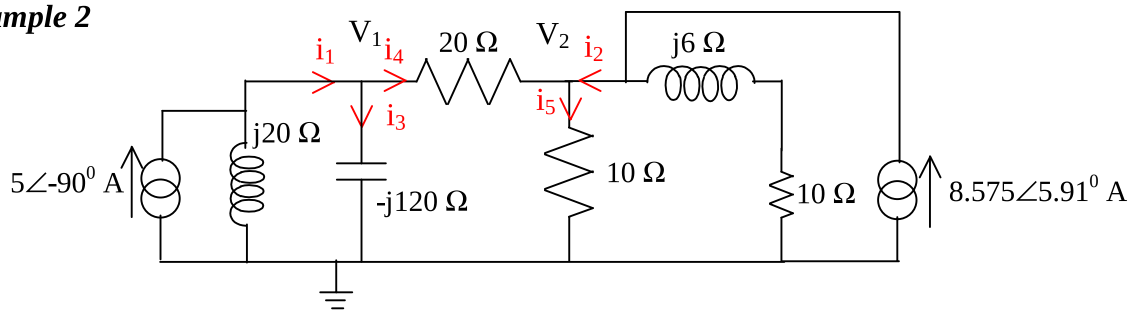
$$I_{gi} = V_i \sum_{\substack{j=1\\ j\neq i}}^{N} y_k - \sum_{\substack{j=1\\ j\neq i}}^{N} y_k V_j = \left(\sum_{\substack{j=1\\ j\neq i}}^{N} y_k \right) V_i + \sum_{\substack{j=1\\ j\neq i}}^{N} (-y_k) V_j$$

$$I_{gi} = y_{ii} V_i + \sum_{\substack{j=1\\ j\neq i}}^{N} y_{ii} V_j = \sum_{j=1}^{N} y_{ij} V_j$$

Thus we see that the final equation derived in this manner actually corresponds to the nodal equation, and that the diagonal term of the matrix actually corresponds to the branch admittances connected to the node, and that the off diagonal terms actually correspond to the negative of the branch admittance.

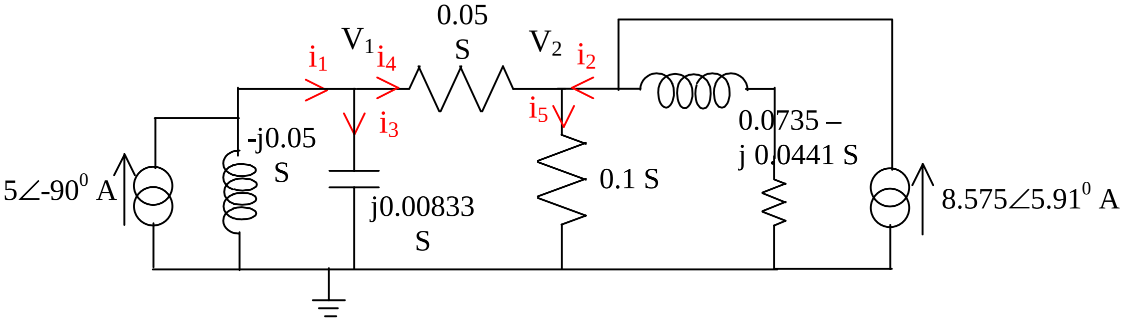
As in the case of mesh analysis, the nodal equation $\underline{I_{gN}} = [Y_N]\underline{V_N}$ is first solved to give $\underline{V_N}$ and the branch voltages and branch currents can then be obtained using the matrix equations.

Example 2



Example 1 has been reformulated as a problem with current sources rather than with voltage sources. [If voltage sources are present, they would first have to be converted to current sources].

The network may also be drawn in terms of admittances as follows.



The branch-node incidence matrix [A], branch injected current $\underline{I_{gb}}$, and the branch admittance matrix may be written, with the reference selected as the earthed node as

$$[A]=\begin{bmatrix}-1 & 0\\ 0 & -1\\ 1 & 0\\ 1 & -1\\ 0 & 1\end{bmatrix},\qquad \underline{I_{gb}}=\begin{bmatrix}5\angle-90^o\\ 8.575\angle5.91^o\\ 0\\ 0\\ 0\end{bmatrix},$$

$$[Y_b]=\begin{bmatrix}-j0.05 & 0 & 0 & 0 & 0\\ 0 & 0.0735-j0.0441 & 0 & 0 & 0\\ 0 & 0 & j0.008333 & 0 & 0\\ 0 & 0 & 0 & 0.05 & 0\\ 0 & 0 & 0 & 0 & 0.1\end{bmatrix}$$

As in the case of mesh analysis, the nodal current injection vector and the nodal admittance matrix may be written from first principles. I leave this as an exercise for you and I will work it by inspection.

$$[I_{gN}]=\begin{bmatrix}5\angle-90^o\\ 8.575\angle5.91^o\end{bmatrix},\quad [Y_N]=\begin{bmatrix}-j0.05+j0.00833+0.05 & -0.05\\ -0.05 & 0.05+0.1+0.0735-j0.0441\end{bmatrix}$$

$$\therefore\begin{bmatrix}5\angle-90^o\\ 8.575\angle5.91^o\end{bmatrix}=\begin{bmatrix}-j0.05+j0.00833+0.05 & -0.05\\ -0.05 & 0.05+0.1+0.0735-j0.0441\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix}$$

$$\therefore\begin{bmatrix}V_1\\ V_2\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix}0.05+0.1+0.0735-j0.0441 & 0.05\\ 0.05 & -j0.05+j0.00833+0.05\end{bmatrix}\begin{bmatrix}5\angle-90^o\\ 8.575\angle5.91^o\end{bmatrix}$$

Δ = (-j0.05+j0.00833+0.05)(0.05+0.1+0.0735-j0.0441) – 0.05^2

= (0.05 – j 0.04167)(0.2235 – j 0.0441) – 0.0025

= 0.06509∠-39.81°×0.2278∠-11.16° – 0.0025

= 0.01483∠-50.97 – 0.0025 = 0.00934 – 0.0025 – j 0.01152 = 0.00684 – j 0.01152

= 0.0134∠-59.30°

∴ V_1 = (0.2278∠-11.16°×5∠-90° + 0.05×8.575∠5.91°)/0.0134∠-59.30°

= (– 0.2205 – j 1.1175 + 0.4265 + j 0.04415) /0.0134∠-59.30°

= (0.2060 – j 1.0733)/0.0134∠-59.30° = 1.093∠-79.14°/0.0134∠-59.30°

$V_1 = \ 81.6\angle\text{-}19.84^o$ V

$\therefore$ branch current $\qquad i_1 = \dfrac{100 - 81.6\angle -19.84^o}{j20} = \dfrac{23.3 + j27.68}{j20}$

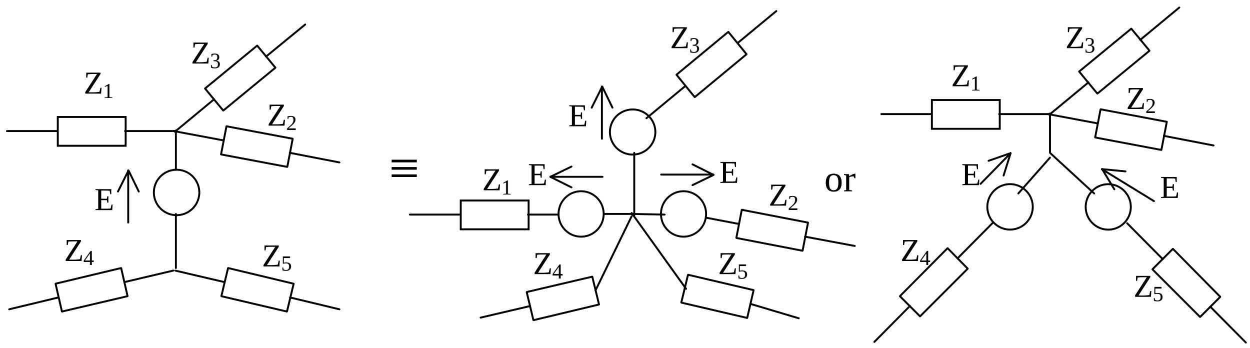
$i_1 \quad = 1.384 - j1.165 = 1.809\angle -40.09^o \ A$

which is the same answer (to calculation accuracy) that was obtained in example 1.

Conversion of Ideal sources

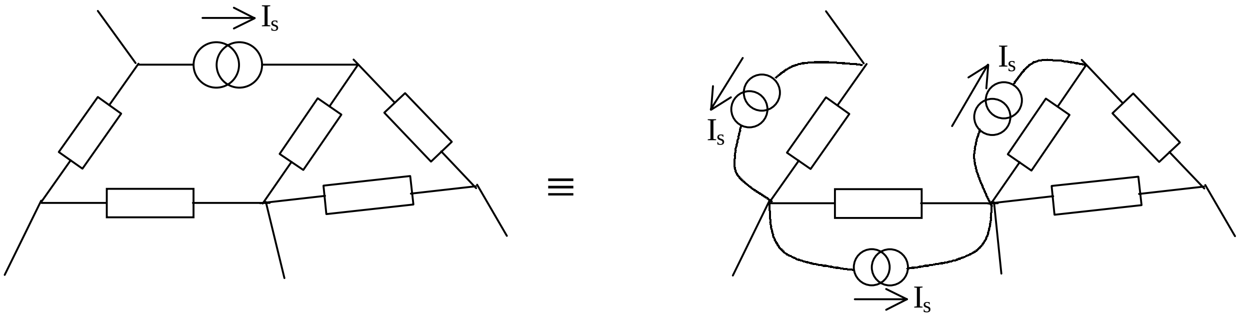
When voltage sources with no series impedance occurs or current sources with no shunt admittance occurs in a network, then no conversion of source type can be made directly. However, a circuit will always have other impedances/admittances so that the following procedure can be adopted, where a single source is replaced by a number of equal sources distributed in the network.

Consider the following circuit where no impedance is directly in series with the voltage source.



In the case of an ideal voltage source, it is distributed to the branches connected to one of the nodes of the original ideal source. Of course with this type of distribution, where there was a common star point earlier, there would be a number of corresponding points in the new network.

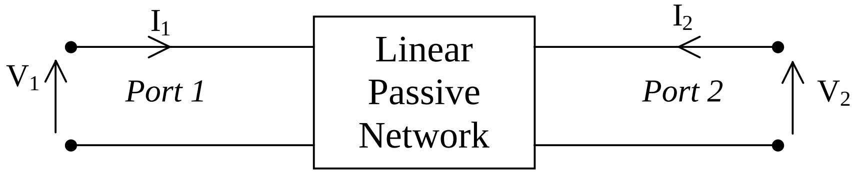
Consider the following circuit where no admittance appears directly in parallel with a current source.



The ideal current source has been distributed around a loop connecting the two points of the original source. Since the current coming into one of the new nodes is the same as the current going out of the node, there is no overall change.

Two-Port Theory

It is convenient to develop special methods for the systematic treatment of networks. In the case of a single port linear active network, we obtained the Thevenin's equivalent circuit and the Norton's equivalent circuit. When a linear passive network is considered, it is convenient to study its behaviour relative to a pair of designated nodes.



Let us learn about a few terms before we proceed with the analysis.

A ***Port*** is a pair of nodes across which a device can be connected. The voltage is measured across the pair of nodes and the current going into one node is the same as the current coming out of the other node in the pair. These pairs are entry (or exit) points of the network.

[Compare with an Airport or a Sea Port. These are entry and exit points to a country. The planes that enter at a given port are the ones that take off from the same port].

The ***Driving point impedance*** is defined as the ratio of the applied voltage (*driving point voltage)* across a node-pair to the current entering at the same port. [This also corresponds to the *input impedance* of the network seen from the particular port.]

Driving point impedance at Port 1 = V_1/I_1

Driving point impedance at Port 2 = V_2/I_2

The ***Driving point admittance*** is similarly defined as the ratio of the current entering at a port to the applied voltage across the same node-pair.

Driving point admittance at Port 1 = I_1/V_1

Driving point admittance at Port 2 = I_2/V_2

Note: The term *Immittance* may sometimes be used to represent either an impedance or an admittance

The ***Transfer impedance*** is defined as the ratio of the applied voltage across a node-pair to the current entering at the other port.

Transfer impedance = V_1/I_2 , V_1/I_2

The ***Driving point admittance*** is similarly defined as the ratio of the current entering at a port to the voltage appearing across the other node-pair.

Transfer admittance = I_1/V_2 , I_2/V_1

The ***Transfer Voltage gain (or ratio)*** is defined as the ratio of the voltage at a node pair to the voltage appearing at the other node-pair.

Transfer voltage = V_1/V_2 , V_2/V_1

The ***Transfer Current gain (or ratio)*** is similarly defined as the ratio of the current at a port to the current at the other port.

Transfer current = I_1/I_2 , I_2/I_1

The external conditions of a two-port network can be completely defined by the currents and voltages at the two ports. Hence a general two port network can be characterised by four parameters, which may be derived from the network elements.

In the case of symmetry, the number of independent parameters will be reduced.

The common parameters used to describe two port networks are

(a) Impedance parameters

(b) Admittance parameters

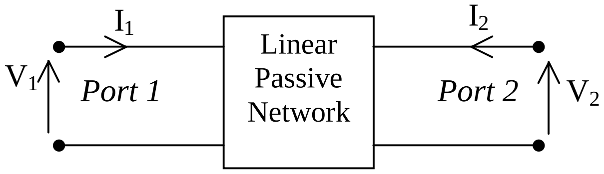
(c) Transmission Line parameters, and

(d) Hybrid parameters.

(a) Impedance Parameters (z-parameters) or Open-circuit parameters

The impedance parameters represent the relation between the voltages and the currents in the two port network.

The impedance parameter matrix may be written as



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

In this matrix equation, it is easily seen without even expanding the individual equations, that

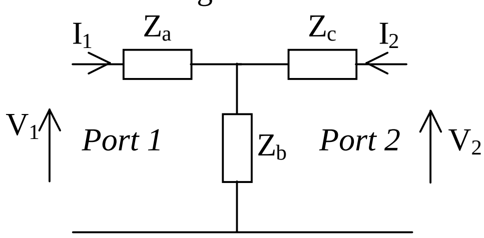
$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}, \quad z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}, \quad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}, \quad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}.$$

It can be seen that the z-parameters correspond to the ***driving point*** and ***transfer*** impedances at each port with the other port having zero current (i.e. open circuit). Thus these parameters are also referred to as the open circuit parameters.

Example 3

Find the impedance parameters of the two port ***T – network*** shown in the figure.

For this case, with port 2 on open circuit



$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = Z_a + Z_b$$

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_b$$

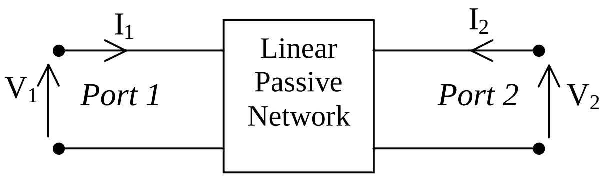
similarly with port 1 open, $z_{12} = Z_b$, $z_{22} = Z_b + Z_c$

Thus the impedance parameter matrix is written as

$$[Z] = \begin{bmatrix} Z_a + Z_b & Z_b \\ Z_b & Z_b + Z_c \end{bmatrix}$$

(b) Admittance Parameters (y-parameters) or Short-circuit parameters

The admittance parameters represent the relation between the currents and the voltages in the two port network.



The admittance parameter matrix may be written as

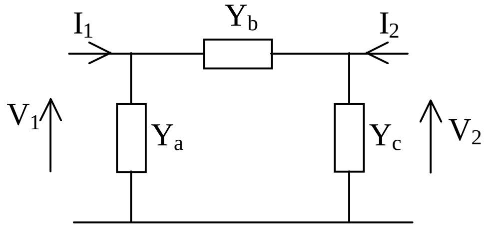
$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

The parameters y_{11}, y_{12}, y_{21}, y_{22} can be defined in a similar manner, with either V_1 or V_2 on short circuit.

It can be seen that the y-parameters correspond to the ***driving point*** and ***transfer*** admittances at each port with the other port having zero voltage (i.e. short circuit). Thus these parameters are also referred to as the short circuit parameters.

Example 4

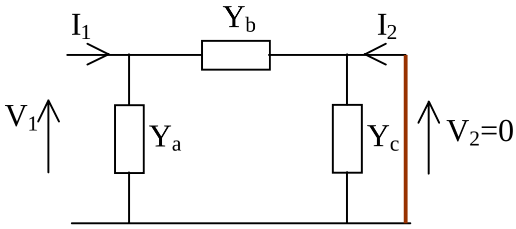
Find the impedance parameters of the 2 port ***π – network*** shown in the figure.



For this case, with port 2 on short circuit

$$y_{11} = \left. \frac{I_1}{V_1} \right|_{V_2=0} = Y_a + Y_b$$

$$y_{21} = \left. \frac{I_2}{V_1} \right|_{V_2=0} = -Y_b$$



similarly with port 1 shorted, $y_{12} = -Y_b$, $y_{22} = Y_b + Y_c$

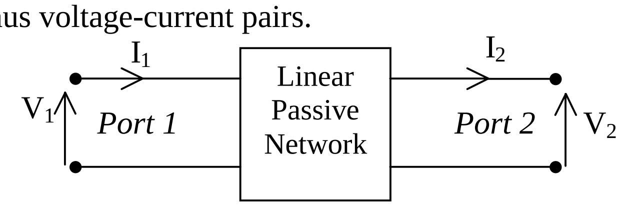
Thus the admittance parameter matrix is written as

$$[Y] = \begin{bmatrix} Y_a + Y_b & -Y_b \\ -Y_b & Y_b + Y_c \end{bmatrix}$$

(c) Transmission Line Parameters (ABCD-parameters)

The ABCD parameters represent the relation between the input quantities and the output quantities in the two port network. They are thus voltage-current pairs.

However, as the quantities are defined as an input-output relation, the output current is marked as going out rather than as coming into the port.



The impedance parameter matrix may be written as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

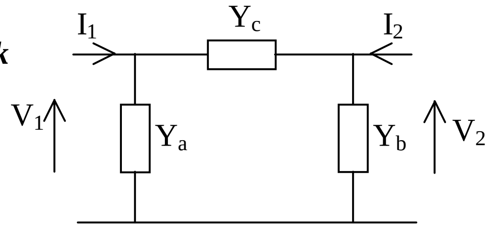
The parameters A, B, C, D can be defined in a similar manner with either port 2 on short circuit or port 2 on open circuit. These parameters are known as transmission parameters as in a transmission line, the currents enter at one end and leaves at the other end, and we need to know a relation between the sending end quantities and the receiving end quantities.

In the case of symmetrical system, such as a transmission line, where the properties from one end are the same as the properties from the other end, parameter A = D.

Also in the case of a reciprocal system, it can be shown that A.D – B.C = 1

Example 5

Find the ABCD parameters of the 2 port ***π – network*** shown in the figure.



For this case, it can be shown that

$$A = \frac{Y_b + Y_c}{Y_c},\ B = \frac{1}{Y_c},\ C = \frac{Y_aY_b + Y_bY_c + Y_cY_a}{Y_c},$$

and $D = \frac{Y_a + Y_c}{Y_c}$

[It can be seen that for a symmetrical network, where $Y_a = Y_c$, A = D].

Also,

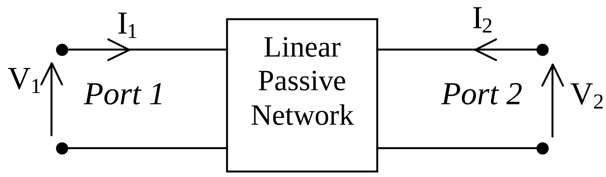
$$A.D - B.C \quad = \frac{Y_b + Y_c}{Y_c}\cdot\frac{Y_a + Y_c}{Y_c} - \frac{1}{Y_c}\cdot\frac{Y_aY_b + Y_bY_c + Y_cY_a}{Y_c}$$

$$= \frac{Y_bY_a + Y_cY_a + Y_bY_c + Y_c^2 - (Y_aY_b + Y_bY_c + Y_cY_a)}{Y_c^2} = 1$$

(d) Hybrid Parameters (h-parameters)

The hybrid parameters represents a mixed or hybrid relation between the voltages and the currents in the two port network.

The hybrid parameter matrix may be written as



$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}\begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

The h-parameters can be defined in a similar manner and are commonly used in some electronic circuit analysis. The method of obtaining the parameters is very similar to the earlier cases.

Interconnection of two-port networks

In certain applications, it becomes necessary to connected the two-port networks together. The common connections are (a) series, (b) parallel and (c) cascade.

(a) Series connection of two-port networks

As in the case of elements, a series connection is defined when the currents in the series elements are equal and the voltages add up to give the resultant voltage.

In the case of two-port networks, this property must be applied individually to each of the ports.

Thus, if we consider 2 networks ***r*** and ***s*** connected in series

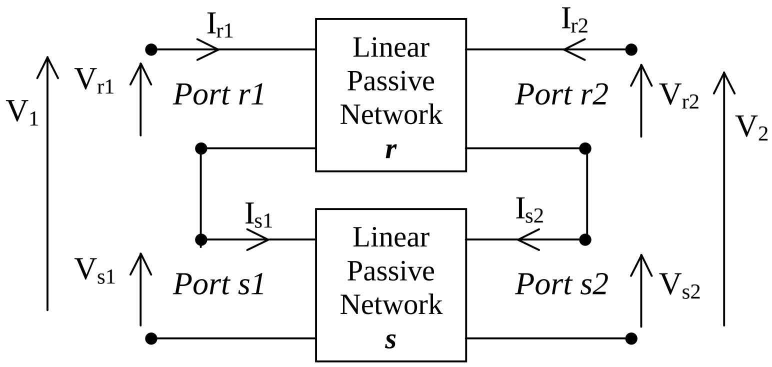
at port 1, $I_{r1} = I_{s1} = I_1$, and $V_{r1} + V_{s1} = V_1$

similarly, at port 2 $I_{r2} = I_{s2} = I_2$ and $V_{r2} + V_{s2} = V_2$

The two networks, ***r*** and ***s*** can be connected in following manner to be in series with each other.

Under these conditions, it can be easily seen that if impedance parameters are used, then the resultant impedance parameter matrix for the series combination is the addition of the two individual impedance matrices.

$[Z] = [Z_r] + [Z_s]$



(b) Parallel connection of two-port networks

As in the case of elements, a parallel connection is defined when the voltages in the parallel elements are equal and the currents add up to give the resultant current.

In the case of two-port networks, this property must be applied individually to each of the ports.

Thus, if we consider 2 networks ***r*** and ***s*** connected in parallel

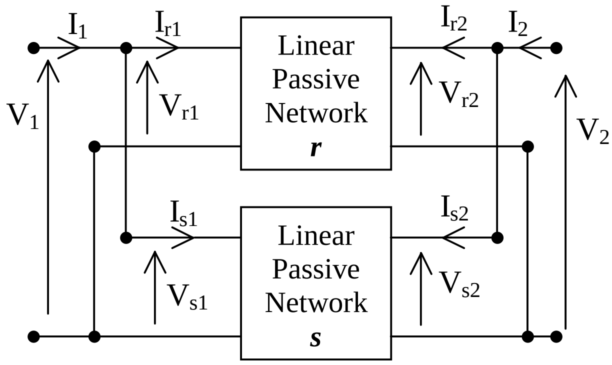
at port 1, $I_{r1} + I_{s1} = I_1$, and $V_{r1} = V_{s1} = V_1$

similarly, at port 2 $I_{r2} + I_{s2} = I_2$ and $V_{r1} = V_{s1} = V_1$

The two networks, ***r*** and ***s*** can be connected in following manner to be in parallel with each other.

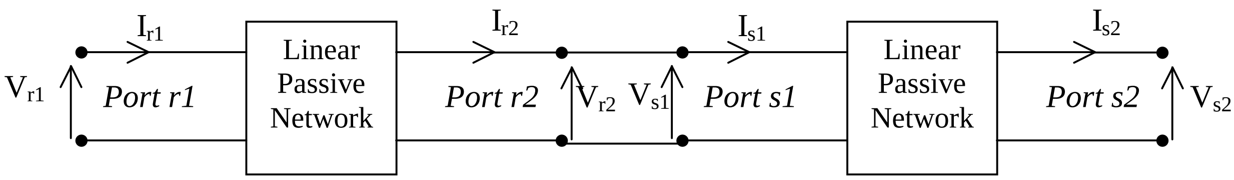
Under these conditions, it can be easily seen that if admittance parameters are used, then the resultant admittance parameter matrix for the parallel combination is the addition of the two individual admittance matrices.

$[Y] = [Y_r] + [Y_s]$



(c) Cascade connection of two-port networks

A cascade connection is defined when the output of one network becomes the input to the next network.



It can be easily seen that $I_{r2} = I_{s1}$ and $V_{r2} = V_{s1}$

Therefore it can easily be seen that the ABCD parameters are the most suitable to be used for this connection.

$$\begin{bmatrix} V_{r1} \\ I_{r1} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \begin{bmatrix} V_{r2} \\ I_{r2} \end{bmatrix}, \quad \begin{bmatrix} V_{s1} \\ I_{s1} \end{bmatrix} = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \begin{bmatrix} V_{s2} \\ I_{s2} \end{bmatrix}$$

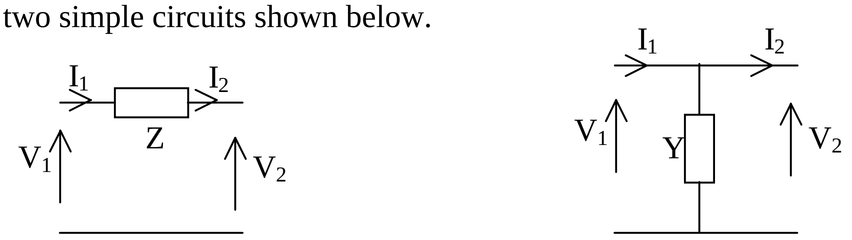
If the intermediate voltage-current pairs are eliminated, the matrix equation can be re-written in the following form, as a product matrix.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

Thus it is seen that the overall ABCD matrix is the product of the two individual ABCD matrices.

This is a very useful property in practice, especially when analysing transmission lines.

Consider the two simple circuits shown below.



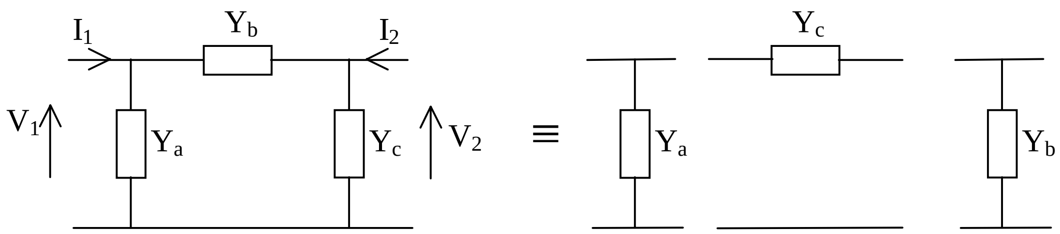
The ABCD matrices for these two networks can be written almost by inspection as follows.

$A = \left.\dfrac{V_1}{V_2}\right\|_{I_2=0}$	$= 1,$	$= 1$
$B = \left.\dfrac{V_1}{I_2}\right\|_{V_2=0}$	$= Z,$	$= 0$
$C = \left.\dfrac{I_1}{V_2}\right\|_{I_2=0}$	$= 0,$	$=Y$
$D = \left.\dfrac{I_1}{I_2}\right\|_{V_2=0}$	$= 1,$	$= 1$

or in matrix form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix}, \qquad = \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix}$$

Using these results, we will now see how the earlier example 5 could be analysed.



You can see that it is composed of three two-port networks in cascade. Thus

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y_a & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/Y_c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_b & 1 \end{bmatrix}$$

Simplification of this matrix product would give the same answer as in example 5.