Basic Circuit Elements - *Prof J R Lucas*

An electrical circuit is an interconnection of electrical circuit elements. These circuit elements can be categorized into two types, namely active elements and passive elements.

Some Definitions/explanations of electrical terms

Charge (unit: *coulomb,* C; letter symbol: q or Q)

The *electric charge* is the most basic quantity in electrical engineering, and arises from the atomic particles of which matter is made.

Potential Difference (unit: *volt,* V; letter symbol: v or V)

The *potential difference*, also known as *voltage*, is the work done (or energy required) to move a unit positive charge from one point to another (across a circuit element).

Thus the change in work done d*w* when a charge *dq* moves through a potential difference of *v*

$$\mathrm{d}w = v.dq$$

Current (unit: *ampere,* A; letter symbol: i or I)

The *electric current* is the rate of charge flow in a circuit.

$$i = \frac{dq}{dt}, \qquad q = \int i.dt$$

Energy (unit: *joule*, J; letter symbol: w or W)

The *Energy* is the capacity to do work.

Thus in electrical quantities this may be expressed as

$$\int \mathrm{d}w = \int v.dq = \int v.i.dt$$

Power (unit: *watt*, W; letter symbol: p or P)

The *electric power* is the rate of change of energy.

$$p = \frac{dw}{dt} = v.i$$

Common usage of letter symbols

It is common practice to use the simple letters (such as *v, i, p, w*) to represent quantities which are varying with time, and capital letters (such as *V, I, P, W*) to represent quantities which are constants. But this need not always be done and is a useful practice rather than a rule. With representation of elements, obviously this practice does not exist as they are not time variables.

The letter *p* is also commonly used to represent the differential operator $\frac{d}{dt}$.

The letter j is normally used for the imaginary operator $\sqrt{-1}$ as i is almost invariably used to denote current.

Passive Circuit Elements

The most basic of the passive circuit elements are the resistance, inductance and capacitance. Passive elements do not generate (convert from non-electrical energy) any electricity. They may either consume energy (i.e. convert from electrical form to a non-electrical form such as heat or light), or store energy (in electrostatic and electromagnetic fields).

Resistance (unit: *ohm,* Ω; letter symbol: *R , r*)

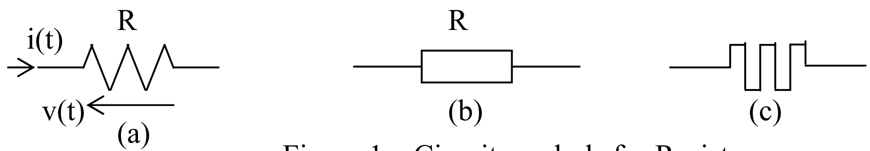

Figure 1 – Circuit symbols for Resistance

The common circuit symbols for the Resistor are shown in figure 1. Figure 1(a) is the common symbol used for the general resistor, especially when hand-written. Figure 1(b) is the most general symbol for the resistor, especially when in printed form. Figure 1(c) is the symbol used for a non-inductive resistor, when it is necessary to clearly indicate that it has been specially made to have no or negligible inductance. A resistor made in coil form, must obviously have at least a small amount of inductance.

The basic equation governing the resistor is Ohm's Law (see also page 5).

$$v(t) = R \,.\, i(t)$$

This may also be written as

$$i(t) = G \,.\, v(t), \qquad G = \frac{1}{R}$$

where G is the conductance (unit: *siemen*, S)

$$p(t) = v(t) \,.\, i(t) = R \,.\, i^2(t) = G \,.\, v^2(t)$$

$$w(t) = \int v(t) \,.\, i(t) \,.\, \mathrm{dt} = \int R \,.\, i^2(t) \,.\, \mathrm{dt} = \int G \,.\, v^2(t) \,.\, \mathrm{dt}$$

It is to be noted that *p(t)* is always positive indicating that power is always consumed and energy always increases with time.

Inductance (unit: *henry,* H; letter symbol: *L , l*)

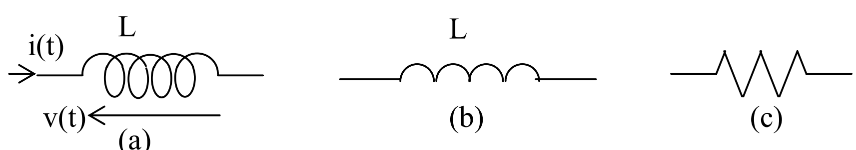

Figure 2 – Circuit symbols for Inductance

The common circuit symbols for the Inductor are shown in figure 2. Figure 2(a) shows a coil which is the simplest symbol (and most common when hand-written) for the inductor. A simpler representation of this is shown in figure 2(b) and is used to simplify the drawing of circuits. The symbol shown in figure 2(c) is sometimes used in printed form, especially on transformer nameplates, but is not a recommended form as it could lead to confusion with the common resistor.

The basic equation governing the behaviour of an inductor is Faraday's law of electromagnetism.

$$e = (-)\frac{d\phi}{dt}$$

When there are N turns in a coil, e.m.f. will be induced in each turn, so that the voltage across the coil would be N times larger. Also, if the voltage is measured as a drop, the negative sign vanishes, so that

$$v = N\frac{d\phi}{dt}$$

The flux produced in the magnetic circuit, is proportional to the current flowing in the coil, so that we may express the rate of change of flux in terms of a rate of change of current.

$$\phi \propto i\,, \quad v \propto \frac{d\,i}{dt}$$

This is written as a basic electrical circuit equation as

$$v = L\frac{d\,i}{dt} = L\,p\cdot i \qquad \text{or} \qquad i = \frac{1}{L}\int i\cdot dt = \frac{1}{L\,p}\cdot i$$

Since the voltage across an inductor is proportional to the rate of change of current, a step current change is not possible through an inductor as this would correspond to an infinite voltage. i.e. the current passing through an inductor can never change suddenly. You might ask, whether this would not occur if we switched off the current in an inductor. What would really happen is that the ensuing high rate of change would cause a very large voltage to develop across the switch, which in turn would cause a spark over across the gap of the switch continuing the current for some more time.

$$p(t) = v(t)\ .\ i(t)$$

$$w(t) = \int v(t)\ .\ i(t)\ .\ dt\ = \int\ L\frac{d\,i}{d\,t}\cdot i\cdot dt = \int L\ .\ i.\mathrm{di}\ =\ \tfrac{1}{2}L.\ i^{2}$$

It can be seen that $w(t)$ is dependant only on i and not on time. Thus when the current i increases, the energy consumed increases and when i decreases, the energy consumed decreases. This actually means that there is no real consumption of energy but storage of energy.

[If we compare with water tap, opening it and letting the water run into the ground would correspond to water consumption, where as filling a bucket with the water, and perhaps putting it back into the water tank, would correspond to water storage].

Thus an inductor does not consume electrical energy, but only stores it in the electromagnetic field.

Stored energy $w(t) = \tfrac{1}{2}L.\ i^{2}$

Capacitance (unit: *farad,* F; letter symbol: C , c)

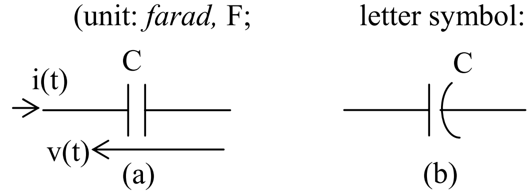

Figure 3 – Circuit symbols for Capacitance

The common circuit symbols for the Capacitor are shown in figure 3.

When a voltage is applied across a capacitor, a positive charge is deposited on one plate and a negative charge on the other and the capacitor is said to store a charge.

The charge stored is directly proportional to the applied voltage.

$$q = C \,.\, v$$

Since $q = \int i \,.\, dt$ the basic equation for the capacitor may be re-written in circuit terms as

$$v = \frac{1}{C}\int i \cdot dt \qquad \text{or} \qquad i = C\frac{d\,v}{d\,t}$$

Since the current through a capacitor is proportional to the rate of change of voltage, a step voltage change is not possible through a capacitor as this would correspond to an infinite current. i.e. the voltage across a capacitor can never change suddenly.

$$p(t) = v(t) \,.\, i(t)$$

$$w(t) = \int v(t) \,.\, i(t) \,.\, dt = \int v \,.\, C\frac{d\,v}{d\,t} \cdot dt = \int C \,.\, v.\mathrm{dv} = \tfrac{1}{2}C.\, v^2$$

It can be seen that *w(t)* is dependant only on *v* and not on time. Thus when the voltage *v* increases, the energy consumed increases and when *v* decreases, the energy consumed decreases. This actually means that there is no real consumption of energy but storage of energy.

Thus a capacitor does not consume electrical energy, but only stores it in the electromagnetic field.

Stored energy $w(t) = \tfrac{1}{2}C.\, v^2$

Summary

For a resistor, $v = R\, i,$ $i = G\, v$

For an inductor, $v = Lp\, i,$ $i = \frac{1}{Lp} v$

Current through an inductor will never change suddenly.

For a capacitor, $v = \frac{1}{Cp} i,$ $i = Cp\, v$

Voltage across a capacitor will never change suddenly.

Impedance and Admittance

These may all be written in the form

$$v = Z(p)\, i\,, \qquad i = Y(p)\, v$$

where *Z(p)* is the impedance operator, and *Y(p)* is the admittance operator.

Impedances and Admittances may be either linear or non-linear. This is defined based on whether the values of R, L and C (slope of characteristic) are constants or not.

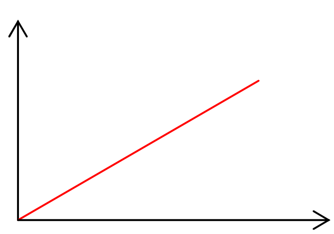

Figure 4(a) Linear System

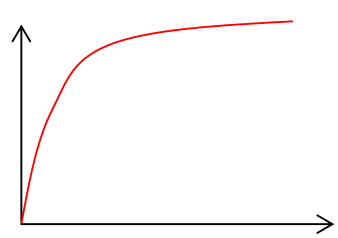

Figure 4(b) Non-Linear System

Ohm's Law

This law was first described by Professor Georg Simon Ohm (1791-1867) in a pamphlet "Die galvanische Kette mathematisch bearbeitet" (*translation* The Galvanic Circuit investigated mathematically) in 1827.

In any conducting medium, the current density $\vec{J}$ is related to the electric field $\vec{E}$ by the conductivity σ of the medium (which can be a tensor in anisotropic materials) as

$\vec{J} = \sigma \vec{E}$ *(vector form of Ohm's Law)*

Note: This form of the equation is only valid in the reference frame of the conducting material. If the material is moving at velocity v relative to a magnetic field B, $J = \sigma.(E + v \times B)$

The total current I flowing across a surface S, perpendicular to it, is given by

$I = \int_S J.dS = J\,.\,S$ for a conducting rod of uniform cross-section

In an electric field *E,* the difference of potential between the ends of a conductor is given as

$V = \int E.dl = E\,.\,l$ for a uniform field

Thus $\frac{I}{S} = \sigma.\frac{V}{l}$, so that $I = \frac{\sigma.S}{l}.V$ and $V = \frac{l}{\sigma.S}.I$

where the constants $\frac{\sigma.S}{l}$ = G = conductance, $\frac{l}{\sigma.S}$ = R = resistance, and $\frac{1}{\sigma} = \rho$ = conductivity

Thus $\boldsymbol{V = R\,.\,I}$ giving the now familiar form of Ohm's Law.

Kirchoff's Laws

Gustav Robert Kirchoff (1824-1887) was the first to publish a systematic formulation of the principles governing the behaviour of electric circuits, based on already available experience. His work embodied two laws, namely a current law and a voltage law.

In a circuit supplied from a battery, external to the battery only an electrostatic field exists. The field ξ may be derived from the gradient of a scalar potential ϕ.

i.e. $\xi = -\nabla\phi$

The line integral of the field through the battery from the negative terminal to the positive terminal, or the voltage rise, is equal to the emf of the battery.

Thus the potential change over a closed loop becomes zero, giving us Kirchoff's voltage law

emf = Σ voltage drop external to circuit

or Σ v = 0 where v is the voltage change across any circuit element.

The net flow of current into a volume V is accompanied by an increase of charge within the same volume V.

$$-\oint_S J.dS = \frac{\partial}{\partial t}\int_V \rho\, dV$$

Applying Gauss's theorem

$$-\int_V \nabla \cdot J\ dV = \frac{\partial}{\partial t}\int_V \rho\, dV \qquad \text{or} \qquad \int_V \left(\nabla \cdot J + \frac{\partial \rho}{\partial t}\right) dV = 0$$

Since it is true for all V, it follows that

$$\left(\nabla \cdot J + \frac{\partial \rho}{\partial t}\right) = 0$$

For steady currents, $\frac{\partial \rho}{\partial t} = 0$ so that $\nabla.J = 0$

Which results in the Kirchoff's current law

Σ i = 0 where i is the current of any circuit element connected to the node.

Active Circuit Elements

An *Active Circuit Element* is a component in a circuit which is capable of producing or generating energy. [Producing energy actually means converting non-electrical form of energy to an electrical form]. Active circuit elements are thus sources of energy (or simply sources) and can be categorised into voltage sources and the current sources. Voltage sources are those that keep their terminal voltage very nearly the same as their internal voltages ($V \approx E$), while current sources keep their terminal currents very nearly the same as their internal currents ($I \approx I_S$). Thus voltage sources would have series impedances which are relatively small, while current sources would have shunt admittances which are relatively small. Ideal voltage sources would have zero internal impedance while ideal current sources would have zero internal admittance.

Sources can also be categorised as being independent sources, where the generated voltage (or current) does not depend on any other circuit voltage or current; and dependent sources, where the generated voltage (or current) depends on another circuit voltage or current. While the circle is used as the circuit symbol for independent sources, the diamond is used as the circuit symbol for dependent sources.

Independent source

For an independent voltage source (or current source), the terminal voltage (or current) would depend only on the loading and the internal source quantity, but not on any other circuit variable.

(Independent) Voltage Sources

An ideal voltage source (figure 5(a)) keeps the voltage across it unchanged independent of load.

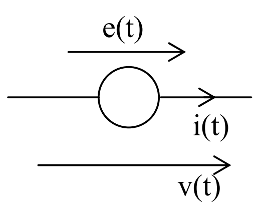

Figure 5(a) – Ideal voltage source

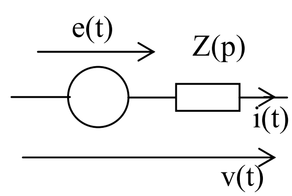

Figure 5(b) – Practical voltage source

$v(t) = e(t)$ for all $i(t)$

$v(t) = e(t) - Z(p).i(t)$

However, practical voltage sources (figure 5(b)) have a drop in voltage across their internal impedances. The voltage drop is generally small compared to the internal emf.

(Independent) Current Sources

An ideal current source (figure 6(a)) keeps the current produced unchanged independent of load.

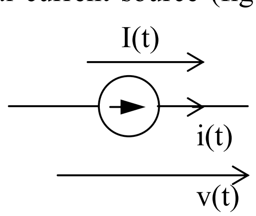

Figure 6(a) – Ideal current source

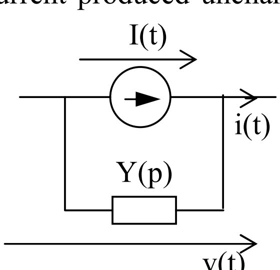

Figure 6(b) – Practical current source

$i(t) = I(t)$ for all $v(t)$

$i(t) = I(t) - Y(p).v(t)$

Practical current sources (figure 6(b)) have a drop in current across their internal admittances. The current drop is generally small compared to the internal source current.

Dependent Source

A dependent voltage source (or current source) would have its terminal voltage (or current) depend on another circuit quantity such as a voltage or current. Thus four possibilities exist. These are (figure 7)

- Voltage dependent (controlled) voltage source
- Current dependent (controlled) voltage source
- Voltage dependent (controlled) current source
- Current dependent (controlled) current source

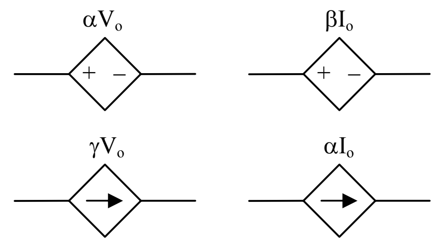

Figure 7 – Dependent sources

Example 1

For the circuit shown in figure 8, determine the current I and V_o.

Answer

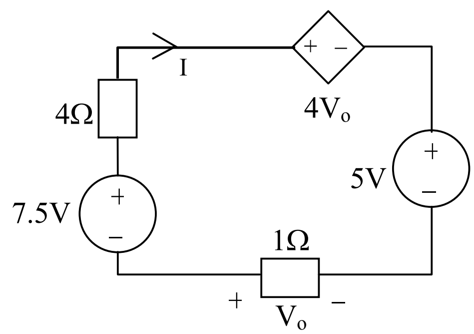

Figure 8 – Circuit for example 1

Applying Kirchoff's voltage law, gives

$$7.5 = 4\,I + 4\,V_o + 5 - V_o$$

Also, from Ohm's law

$$1.I = -\,V_o$$

Thus by substitution, we have

$$I = 2.5\text{ A} \quad \text{and} \quad V_o = -\,2.5\text{ V}$$

Example 2

For the circuit shown in figure 9, determine the current I and V_o.

Answer

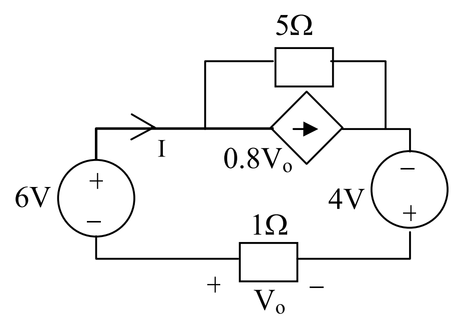

Figure 9 – Circuit for example 2

Applying Kirchoff's voltage law, gives

$$6 = 5\,(I - 0.8\,V_o) - 4 - V_o$$

Also, from Ohm's law

$$1.I = -\,V_o$$

Thus by substitution, we have

$$I = 1.0\text{ A} \quad \text{and} \quad V_o = -\,1\text{ V}$$

Example 3

For the circuit shown in figure 10, determine the current I.

Answer

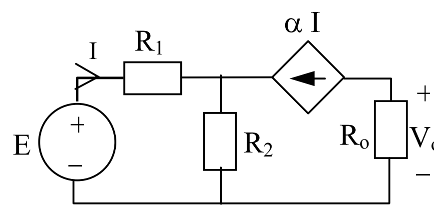

Figure 10 – Circuit for example 3

Applying Kirchoff's voltage law, gives

$$E = R_1\,I + R_2\,(1{+}\alpha)\,I$$

Also, from Ohm's law

$$R_o\,\alpha I = -\,V_o$$

By solution of the equations

$$I = \frac{E}{R_1 + R_2(1+\alpha)}$$

And $$V_o = -\frac{\alpha R_o E}{R_1 + R_2(1+\alpha)}$$

Operational Amplifier (Op Amp)

An operational amplifier is an active circuit element that behaves as a voltage-controlled voltage source. An operational amplifier can be used to add, subtract, multiply, divide, amplify, integrate and differentiate signals and are thus very versatile.

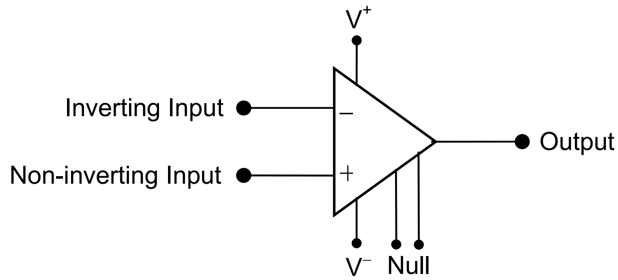

Figure 11 – Circuit connections of Op Amp

A practical Op Amp, commercially available in Integrated circuit (IC) packages would have the inputs and outputs as shown in figure 11. The 'Null' determines the offset.

The output voltage of the Op Amp is linearly proportional to the voltage difference between the input terminals + and – by the gain A. However, the output voltage is limited to the range V^+ to V^- of the supply voltage.

This range is often called the linear region of the amplifier, and when the output reaches to these limits, the op amp is said to be saturated.

The equivalent circuit of the Op Amp is shown in figure 12. It has a dependent voltage source AV_d. An ideal Op Amp has infinite gain ($A = \infty$), infinite input resistance ($R_{in} = \infty$), and zero output resistance ($R_{out} = 0$).

Parameter	Symbol	Ideal	Typical
Open-loop gain	A	∞	10^5 to 10^8
Input Resistance	R_{in}	∞	10^6 to 10^{13} Ω
Output Resistance	R_{out}	0	10 to 100 Ω
Supply Voltage	V^+, V^-		5 to 24 V

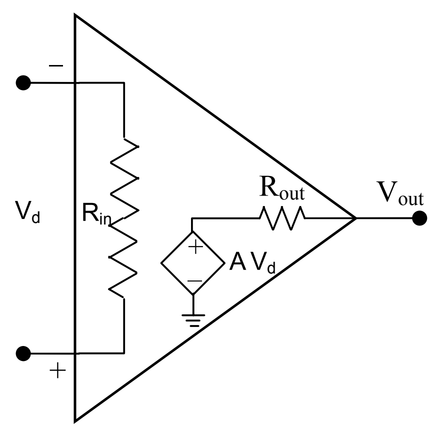

Figure 12 – Equivalent Circuit of Op Amp

Inverting Amplifier

An inverting amplifier circuit is shown in figure 13. In an inverting amplifier the output voltage decreases when the input voltage increases and vice versa.

The equivalent circuit of the inverting amplifier is as shown in the figure 14.

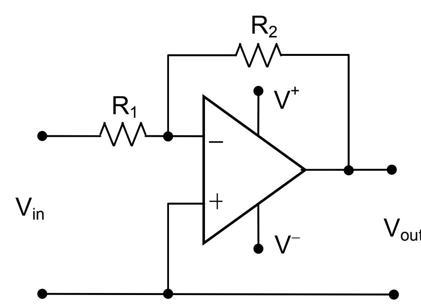

Figure 13 – Circuit of Inverting Amplifier

$$i = \frac{V_{in} - V_d}{R_1} = \frac{V_d}{R_{in}} + \frac{V_d - V_{out}}{R_2}$$

$$\frac{V_d - V_{out}}{R_2} = \frac{V_{out} + AV_d}{R_{out}}$$

If $R_{out} = 0$, and $R_{in} = \infty$,

$V_{out} = -A\ V_d$

$$\frac{AV_{in} - AV_d}{R_1} = \frac{AV_d + A^2 V_d}{R_2}$$

giving $$-\frac{AV_{in}}{R_1} = \frac{V_{out} + AV_{out}}{R_2} + \frac{V_{out}}{R_1}$$

Thus $$\frac{V_{out}}{V_{in}} = -\frac{AR_2}{(1+A)R_1 + R_2}$$

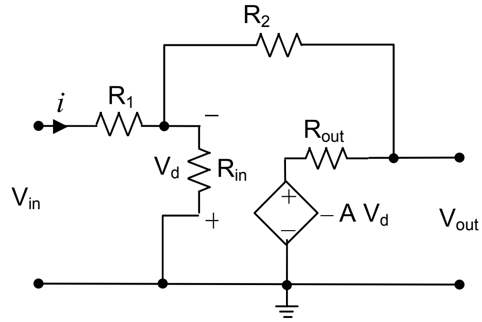

Figure14 – Equivalent Circuit

If A $\rightarrow \infty$ as for an ideal Op Amp,

$$Gain = \frac{V_{out}}{V_{in}} = -\frac{R_2}{R_1}$$

$$V_d = -\frac{V_{out}}{A} \rightarrow 0$$

Thus in ideal Op Amps, V_d is usually taken as a virtual earth, even when analyzing.

Non-inverting Amplifier

In the non-inverting amplifier (figure 15), the input voltage is applied directly to the non-inverting (+) input and a small part of the output voltage is applied to the inverting (–) input from the R_1R_2 potential divider.

For an ideal Op Amp, with $R_{out} = 0$, $R_{in} = \infty$ and $A = \infty$ it can be shown that

$$Gain = \frac{R_1 + R_2}{R_1}$$

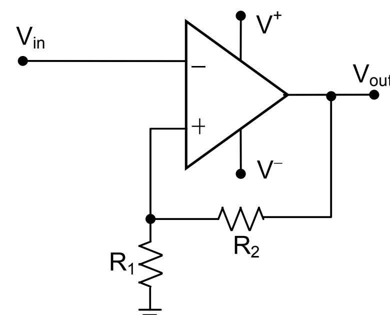

Figure 15 – Non-inverting Amplifier

Summing Amplifier

If another input resistor, equal to the value of the original input resistor R_1, is added to the input of the Op Amp, a Summing amplifier is obtained (figure 16).

$$V_{out} = -\frac{R_2}{R_1}(V_{inA} + V_{inB})$$

If the two input resistors are not equal

$$V_{out} = -R_2\left(\frac{V_{inA}}{R_{1A}} + \frac{V_{inB}}{R_{1B}}\right)$$

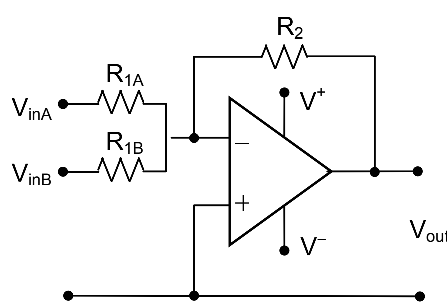

Figure 16 – Summing Amplifier

Differential Amplifier

So far, only one input terminal has been considered, either inverting or non-inverting. It is also possible to connect input signals to both terminals at the same time. The resultant output voltage is proportional to the difference between the two input signals V_{1A} and V_{1B} when the resistance values are appropriately chosen. This type of Op-Amp circuit is commonly known as the Differential Amplifier (figure 17).

For this configuration,

for an ideal Op Amp, $R_{in} = \infty, R_{out} = 0,$ and $A = \infty$.

$A = \frac{V_{out}}{V_d} = \infty$, giving $V_d = \frac{V_{out}}{\infty} = 0$ but with no current going into the Op Amp as $R_{in} = \infty$

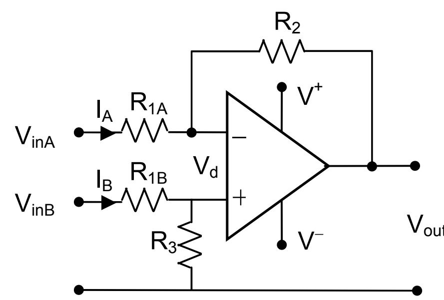

Figure 17 – Differential Amplifier

Thus if the voltage at each input terminal is V,

$$I_A = \frac{V_{inA} - V}{R_{1A}} = \frac{V - V_{out}}{R_2}, \; I_B = \frac{V_{inB} - V}{R_{1B}} = \frac{V}{R_3}$$

This gives $V = \frac{R_3}{R_{1B} + R_3} V_{inB}$

Substitution gives $\frac{V_{out}}{R_2} = V\left(\frac{1}{R_{1A}} + \frac{1}{R_2}\right) - \frac{V_{inA}}{R_A}$

$$V_{out} = \left(\frac{R_{1A} + R_2}{R_{1A}}\right)\left(\frac{R_3}{R_{1B} + R_3}\right)V_{inB} - \frac{R_2}{R_{1A}}V_{inA}$$

When $R_{1A} = R_{1B} = R_1$ and $R_3 = R_2$, the transfer function becomes

$$V_{out} = \frac{R_2}{R_1}\left(V_{inB} - V_{inA}\right)$$

Natural Behaviour of R-L-C Circuits

The natural behaviour of a circuit does not depend on the external forcing functions, but on the system itself. [It is like, if we take a pendulum and give it an initial swing, and then let go, the behaviour of the pendulum depends only on its natural frequency. However, if we keep on pushing it at some other frequency, then the behaviour would also depend on the frequency of the forcing function]. Thus in order to determine the natural behaviour, we must use a forcing function which does not have its own frequency. The two forcing functions that lend themselves to purpose are the step function and the impulse function.

Unit Step Function

The unit step H(t) (similar in appearance to a step in a staircase) has an amplitude zero before time zero, and an amplitude unity after time zero.

$H(t) = 0, \quad t < 0$

$H(t) = 1, \quad t > 0$

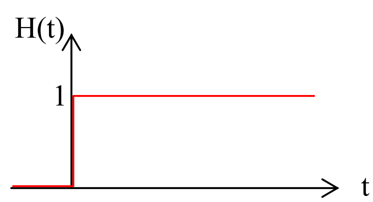

Figure 18 – Unit Step

Unit Impulse Function

The unit impulse δ(t) has an amplitude zero before time zero, and an amplitude of infinity at time zero, and zero again after time zero. It also has the property that the area under the curve is unity.

$$\delta(t) = 0, \quad t < 0$$
$$\delta(t) = \infty, \quad t = 0$$
$$\delta(t) = 0, \quad t > 0$$

also, $\int_{-\infty}^{\infty} \delta(t)\,.\,dt = 1$, which gives $\int_{0^-}^{0^+} \delta(t)\,.\,dt = 1$

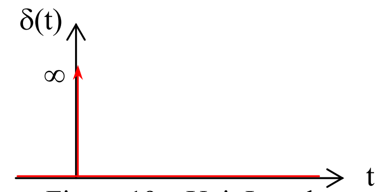

Figure 19 – Unit Impulse

The unit impulse function also has the following properties.

$$\int_{-\infty}^{\infty} f(t)\cdot\delta(t)\,.\,dt = f(0), \text{ and } \int_{-\infty}^{\infty} f(t-\tau)\cdot\delta(t)\,.\,dt = f(\tau)$$

Series R-L circuit

Consider the excitation of a series R-L circuit by a step excitation e(t) = E.H(t).

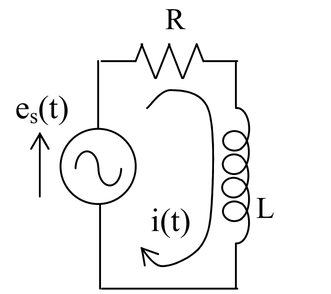

Figure 20 – Series R-L circuit

We can write the differential equation governing the behaviour as

$$L\frac{di}{dt} + R.i = e_s(t) = E.H(t)$$

This equation has a particular integral of E/R corresponding to steady state, and a complementary function corresponding to

$$L\,p\,i + R\,i = 0$$

i.e. the solution to the equation is of the form

$$i(t) = A.e^{-\frac{R}{L}t} + \frac{E}{R}$$

The constant A can be determined from the initial conditions.

i.e. at $t = 0,\ i = 0 \qquad \therefore\ A = (-)\frac{E}{R} \qquad \therefore\ i(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$

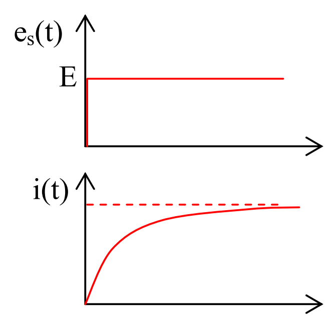

Figure 21 – Step Response

Consider now the excitation of the series R-L circuit by an impulse excitation e(t) = E.δ(t)

The complementary function is the same as before, but the particular integral is now different and equal to 0. The new coefficient A can be obtained from the initial conditions which are now different. The response to a unit impulse is also the same as the derivative of the response to the unit step.

Thus the unit impulse response is

$$i(t) = \frac{d}{dt}\left[\frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right] = \frac{E}{L}e^{-\frac{R}{L}t}$$

Other circuits can also be similarly solved by writing the differential equations governing the behaviour. Further information is available on the section on problems on circuit transients.