

Alternating Current Theory - J R Lucas

An alternating waveform is a periodic waveform which alternate between positive and negative values. Unlike direct waveforms, they cannot be characterised by one magnitude as their amplitude is continuously varying from instant to instant. Thus various forms of magnitudes are defined for such waveforms.

The advantage of the alternating waveform for electric power is that it can be stepped up or stepped down in potential easily for transmission and utilisation. Alternating waveforms can be of many shapes. The one that is used with electric power is the sinusoidal waveform. This has an equation of the form

$$v(t) = V_m \sin(\omega t + \phi)$$

If the period of the waveform is T, then its angular frequency ω corresponds to $\omega T = 2\pi$.

(a) **Instantaneous value:** The instantaneous value of a waveform is the value of the waveform at any given instant of time. It is a time variable $a(t)$.

For a sinusoid, Instantaneous value $a(t) = A_m \sin(\omega t + \phi)$

(b) **Peak value:** The peak value, or *maximum value*, of a waveform is the maximum instantaneous value of the waveform.

For a sinusoid, Peak value = A_m

(c) **Mean value:** The mean value of a waveform is equal to the mean value over a complete cycle of the waveform. It also corresponds to the *direct component* of the waveform.

$$\text{Mean value } A_{\text{mean}} = \frac{1}{T} \int_{t_0}^{t_0+T} a(t).dt$$

The mean value of a waveform which has equal positive and negative half cycles must thus be always zero.

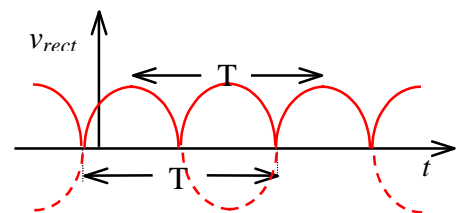
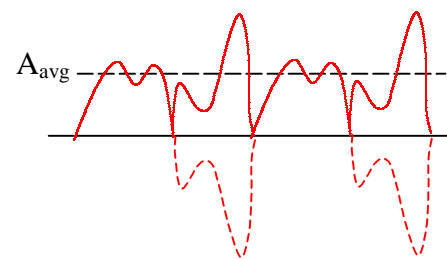
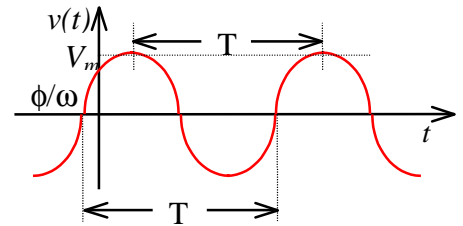
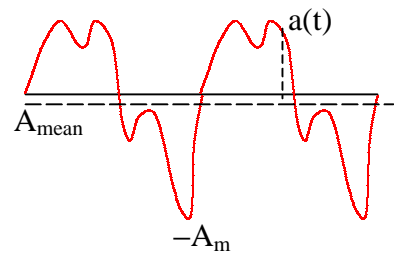
For a sinusoid, Mean value = $\frac{1}{T} \int_{t_0}^{t_0+T} A_m \sin(\omega t + \phi).dt = 0$

(d) **Average value (rectified):** The full-wave rectified average value or *average value* of a waveform is defined as the mean value of the rectified waveform over a complete cycle.

$$\text{Average value } A_{\text{avg}} = \frac{1}{T} \left[\int_{\text{positive hal fcycle}} a(t).dt - \int_{\text{negative hal fcycle}} a(t).dt \right]$$

For a sinusoid,

$$\begin{aligned} \text{Average value} &= \frac{1}{T} \left[\int_0^{T/2} A_m \sin \omega t .dt - \int_{T/2}^T A_m \sin \omega t .dt \right] \\ &= \frac{1}{\omega T} \cdot 2A_m = \frac{2}{\pi} A_m \end{aligned}$$



The average value is defined in the above manner in electrical engineering as otherwise all alternating waveforms would have zero value and would be indistinguishable.

(e) Effective value or r.m.s. value:

Neither the peak value, nor the mean value, nor the average value defines the useful value of the waveform with regard to the power or energy. Thus the effective value is defined based on the power equivalent of the quantity.

The *effective value* is thus defined as the constant value which would cause the same power dissipation as the original quantity over one complete period.

Thus considering current Power dissipation = $I_{eff}^2 \cdot R = \frac{1}{T} \int_{t_o}^{t_o+T} i^2(t) \cdot R \cdot dt$

giving
$$I_{eff} = \sqrt{\frac{1}{T} \int_{t_o}^{t_o+T} i^2(t) \cdot dt}$$

or in general Effective value $A_{eff} = \sqrt{\frac{1}{T} \int_{t_o}^{t_o+T} a^2(t) \cdot dt}$

It is to be noted that the method of calculating the effective value involves the following processes. Taking the **root** of the **mean** of the **squared** waveform over one complete cycle.

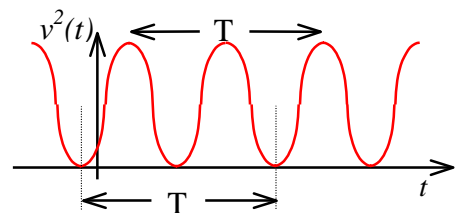
Thus it is also designated as the **root mean square** value, or the **r.m.s. value** of the waveform.

i.e. r.m.s. value $A_{rms} = \sqrt{\frac{1}{T} \int_{t_o}^{t_o+T} a^2(t) \cdot dt}$

This is defined for both current as well as voltage.

For a sinusoid,

r.m.s. value =
$$\sqrt{\frac{1}{T} \int_{t_o}^{t_o+T} A_m^2 \sin^2(\omega t + \phi) \cdot dt} = \frac{A_m}{\sqrt{2}}$$



Unless otherwise specified, the rms value is the value that is always specified for ac waveforms, whether it be a voltage or a current. For example, **230 V** in the mains supply is an **rms value** of the voltage. Similarly when we talk about a **5 A**, **13 A** or **15A** socket outlet (plug point), we are again talking about the **rms value** of the rated current of the socket outlet.

For a given waveform, such as the *sinusoid*, the *peak value*, *average value* and the *rms value* are dependant on each other. The **peak factor** and the **form factor** are the two factors that are most commonly defined.

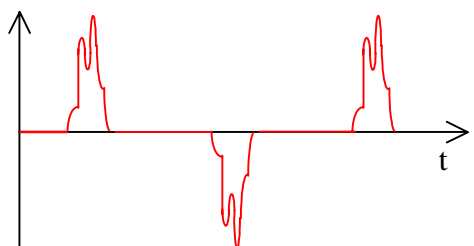
Form Factor =
$$\frac{\text{rms value}}{\text{average value}}$$

and for a sinusoidal waveform, Form Factor =
$$\frac{V_m}{\sqrt{2}} \bigg/ \frac{2V_m}{\pi} = 1.1107 \cong 1.111$$

The form factor is useful such as when the average value has been measured using a rectifier type moving coil meter and the rms value is required to be found. [Note: You will be studying about these meters later]

Peak Factor =
$$\frac{\text{peak value}}{\text{rms value}}$$
 and for a sinusoidal waveform, i

Peak Factor =
$$\frac{V_m}{\frac{V_m}{\sqrt{2}}} = \sqrt{2} = 1.4142$$



The peak factor is useful when defining highly distorted waveforms such as the current waveform of compact fluorescent lamps.

Some advantages of the sinusoidal waveform for electrical power applications

- a. Sinusoidally varying voltages are easily generated by rotating machines
- b. Differentiation or integration of a sinusoidal waveform produces a sinusoidal waveform of the same frequency, differing only in magnitude and phase angle. Thus when a sinusoidal current is passed through (or a sinusoidal voltage applied across) a resistor, inductor or a capacitor a sinusoidal voltage waveform (or current waveform) of the same frequency, differing only in magnitude and phase angle, is obtained.

If $i(t) = I_m \sin(\omega t + \phi)$,

for a resistor, $v(t) = R \cdot i(t) = R \cdot I_m \sin(\omega t + \phi) = V_m \sin(\omega t + \phi)$

– magnitude changed by R but no phase shift

for an inductor,

$$v(t) = L \cdot \frac{di}{dt} = L \cdot \frac{d(I_m \sin(\omega t + \phi))}{dt} = L \cdot \omega \cdot I_m \cos(\omega t + \phi) = L \cdot \omega \cdot I_m \sin(\omega t + \phi + \pi/2)$$

– magnitude changed by $L\omega$ and phase angle changed by $\pi/2$

for a capacitor,

$$v(t) = \frac{1}{C} \cdot \int i \cdot dt = \frac{1}{C} \int I_m \sin(\omega t + \phi) \cdot dt = \frac{-1}{C \cdot \omega} \cdot I_m \cos(\omega t + \phi) = \frac{1}{C \cdot \omega} \cdot I_m \sin(\omega t + \phi - \pi/2)$$

– magnitude changed by $1/C\omega$ and phase angle changed by $-\pi/2$

- c. Sinusoidal waveforms have the property of remaining unaltered in shape and frequency when other sinusoids having the same frequency but different in magnitude and phase are added to them.

$$\begin{aligned} A \sin(\omega t + \alpha) + B \sin(\omega t + \beta) &= A \sin \omega t \cdot \cos \alpha + A \cos \omega t \cdot \sin \alpha + B \sin \omega t \cdot \cos \beta + B \cos \omega t \cdot \sin \beta \\ &= (A \cdot \cos \alpha + B \cdot \cos \beta) \sin \omega t + (A \cdot \sin \alpha + B \cdot \sin \beta) \cos \omega t \\ &= C \sin(\omega t + \theta), \end{aligned}$$

where $C = \sqrt{A^2 + B^2 + 2AB \cos(\alpha - \beta)}$, and $\theta = \tan^{-1} \frac{A \cos \alpha + B \sin \alpha}{A \sin \alpha + B \cos \alpha}$ are constants.

- d. Periodic, but non-sinusoidal waveforms can be broken up to direct terms, its fundamental and harmonics.

$$f(t) = F_0 + F_1 \sin(\omega t + \theta_1) + F_2 \sin(2\omega t + \theta_2) + F_3 \sin(3\omega t + \theta_3) + F_4 \sin(4\omega t + \theta_4) + \dots$$

where F_n and θ_n are constants dependant on the function $f(t)$.

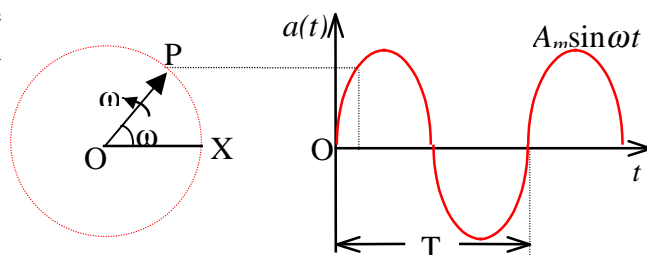
- e. Sinusoidal waveforms can be represented by the projections of a rotating phasor.

Phasor Representation of Sinusoids

You may be aware that $\sin \theta$ can be written in terms of exponentials and complex numbers.

i.e. $e^{j\theta} = \cos \theta + j \sin \theta$

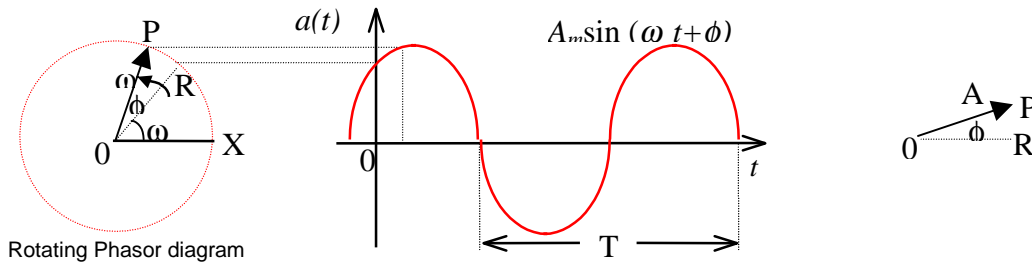
or $e^{j\omega t} = \cos \omega t + j \sin \omega t$



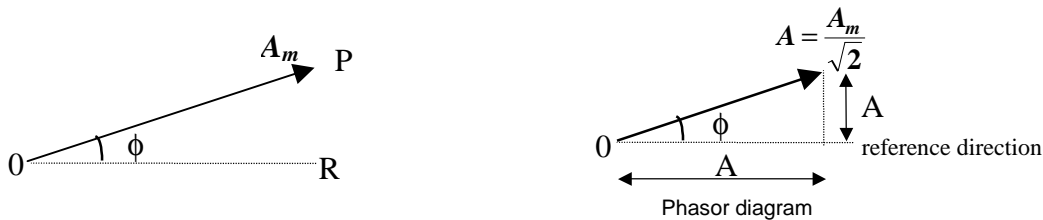
Consider a line OP of length A_m which is in the horizontal direction OX at time $t=0$.

If OP rotates at an angular velocity ω , then in time t its position would correspond to an angle of ωt .

The projection of this rotating phasor OP (a phasor is somewhat similar to a vector, except that it does not have a physical direction in space but a *phase* angle) on the y-axis would correspond to $OP \sin \omega t$ or $A_m \sin \omega t$ and on the x-axis would correspond to $A_m \cos \omega t$. Thus the sinusoidal waveform can be thought of being the projection on a particular direction of the complex exponential $e^{j\omega t}$.



If we consider more than one phasor, and each phasor rotates at the same angular frequency, then there is no relative motion between the phasors. Thus if we fix the reference phasor OR in a particular reference direction (without showing its rotation), then all other phasors moving at the same angular frequency would also be fixed at a relative position. Usually this reference direction is chosen as horizontal on the diagram for convenience.



It is also usual to draw the **Phasor diagram** using the *rms value* A of the sinusoidal waveform, rather than with the *peak value* A_m . This is shown on an enlarged diagram. Thus unless otherwise specified it is the rms value that is drawn on a phasor diagram.

It should be noted that the values on the phasor diagram are no longer time variables. The phasor \mathbf{A} is characterised by its magnitude $|\mathbf{A}|$ and its phase angle ϕ . These are also the polar co-ordinates of the phasor and is commonly written as $|\mathbf{A}| \angle \phi$. The phasor \mathbf{A} can also be characterised by its cartesian co-ordinates A_x and A_y and usually written using complex numbers as $\mathbf{A} = A_x + j A_y$.

Note: In electrical engineering, the letter j is always used for the complex operator $\sqrt{-1}$ because the letter i is regularly used for electric current.

It is worth noting that $|\mathbf{A}| = \sqrt{A_x^2 + A_y^2}$ and that $\tan \phi = \frac{A_y}{A_x}$ or $\phi = \tan^{-1} \left(\frac{A_y}{A_x} \right)$

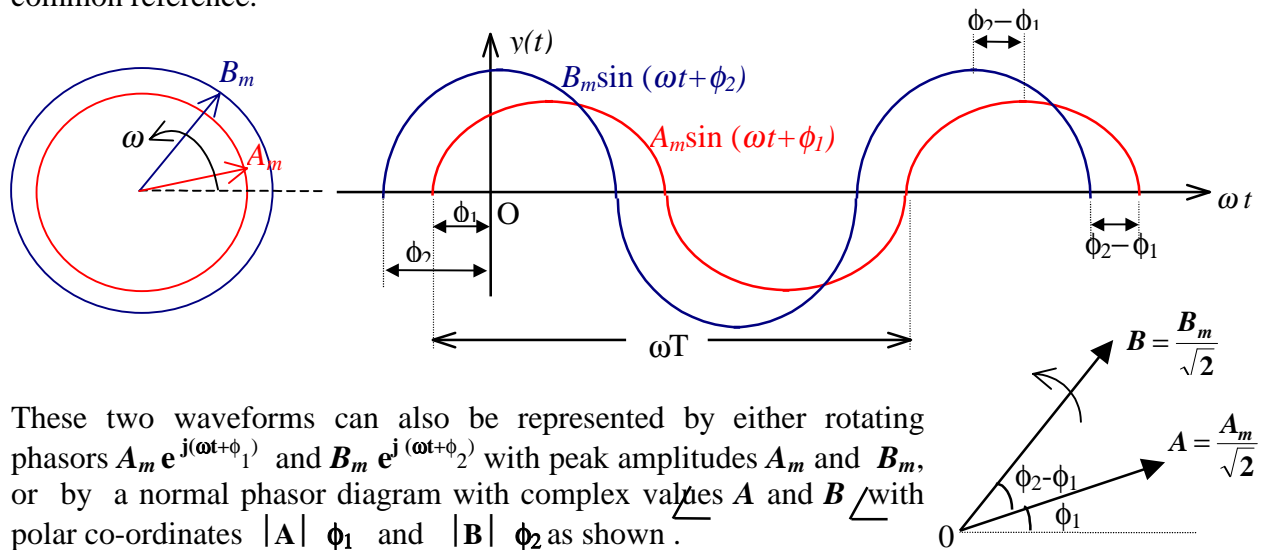
Also, $A_x = |\mathbf{A}| \cos \phi$, $A_y = |\mathbf{A}| \sin \phi$ and $|\mathbf{A}| e^{j\phi} = |\mathbf{A}| \cos \phi + j |\mathbf{A}| \sin \phi = A_x + j A_y$

Note: If the period of a sinusoidal waveform is T , then the corresponding angle would be ωT . Also, the period of a waveform corresponds to 1 complete cycle or 2π radians or 360° .

$$\therefore \omega T = 2\pi$$

Phase difference

Consider the two waveforms $A_m \sin(\omega t + \phi_1)$ and $B_m \sin(\omega t + \phi_2)$ as shown in the figure. It can be seen that they have different amplitudes and different phase angles with respect to a common reference.



These two waveforms can also be represented by either rotating phasors $A_m e^{j(\omega t + \phi_1)}$ and $B_m e^{j(\omega t + \phi_2)}$ with peak amplitudes A_m and B_m , or by a normal phasor diagram with complex values A and B with polar co-ordinates $|A| \angle \phi_1$ and $|B| \angle \phi_2$ as shown.

Any particular value (such as positive peak, or zero) of $a(t)$ is seen to occur at a time T after the corresponding value of $b(t)$. i.e. the positive peak A_m occurs after an angle $(\phi_2 - \phi_1)$ after the positive peak B_m . Similarly the zero of $a(t)$ occurs after an angle $(\phi_2 - \phi_1)$ after the corresponding zero of $b(t)$. In such a case we say that the waveform $b(t)$ **leads** the waveform $a(t)$ by a phase angle of $(\phi_2 - \phi_1)$. Similarly we could say that the waveform $a(t)$ **lags** the waveform $b(t)$ by a phase angle of $(\phi_2 - \phi_1)$. [Note: Only the angle less than 180° is used to specify whether a waveform leads or lags another waveform].

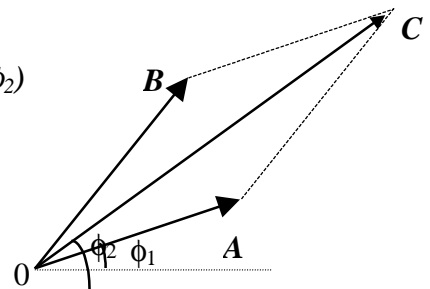
We could also define, **lead** and **lag** by simply referring to the phasor diagram. Since angles are always measured anticlockwise (convention), we can see from the phasor diagram, that B **leads** A by an angle of $(\phi_2 - \phi_1)$ anticlockwise or that A **lags** B by an angle $(\phi_2 - \phi_1)$.

Addition and subtraction of phasors can be done using the same parallelogram and triangle laws as for vectors, generally using complex numbers. Thus the addition of phasor A and phasor B would be

$$\begin{aligned} A + B &= (A \cos \phi_1 + j A \sin \phi_1) + (B \cos \phi_2 + j B \sin \phi_2) \\ &= (A \cos \phi_1 + B \cos \phi_2) + j (A \sin \phi_1 + B \sin \phi_2) \\ &= C_x + j C_y = |C| \angle \phi_c = C \end{aligned}$$

where $|C| = \sqrt{C_x^2 + C_y^2} = \sqrt{(A \cos \phi_1 + B \cos \phi_2)^2 + (A \sin \phi_1 + B \sin \phi_2)^2}$

and $\phi_c = \tan^{-1} \left(\frac{C_y}{C_x} \right) = \tan^{-1} \left(\frac{A \sin \phi_1 + B \sin \phi_2}{A \cos \phi_1 + B \cos \phi_2} \right)$



Example 1

Find the addition and subtraction of the 2 complex numbers given by $10 \angle 30^\circ$ and $25 \angle 48^\circ$.

$$\begin{aligned} \text{Addition} &= 10 \angle 30^\circ + 25 \angle 48^\circ = 10(0.8660 + j 0.5000) + 25(0.6691 + j 0.7431) \\ &= (8.660 + 16.728) + j (5.000 + 18.577) = 25.388 + j 23.577 = 34.647 \angle 42.9^\circ \end{aligned}$$

$$\begin{aligned} \text{Subtraction} &= 10 \angle 30^\circ - 25 \angle 48^\circ = (8.660 - 16.728) + j (5.000 - 18.577) \\ &= -8.068 - j 13.577 = 15.793 \angle 239.3^\circ \end{aligned}$$

Multiplication and division of phasors is most easily done using the polar form of complex numbers.

Thus the multiplication of phasor **A** and phasor **B** would be

$$A * B = |A| \angle \phi_1 * |B| \angle \phi_2 = |A| e^{j\phi_1} * |B| e^{j\phi_2} = |A| * |B| e^{j(\phi_1 + \phi_2)} = |A| * |B| \angle (\phi_1 + \phi_2) = |C| \angle \phi_c$$

where $|C| = |A| * |B|$ and $\phi_c = \phi_1 + \phi_2$

In a similar way, it can be easily seen that for division

$$|C| = |A| / |B| \quad \text{and} \quad \phi_c = \phi_1 - \phi_2$$

Thus, whenever we need to do addition and subtraction, we use the cartesian form of complex numbers, whereas for multiplication or division we use the polar form.

Example 2

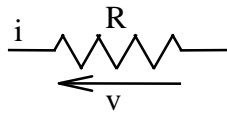
Find the multiplication and the division of the two complex numbers given by $10 \angle 30^\circ$ and $25 \angle 48^\circ$.

Multiplication = $10 \angle 30^\circ * 25 \angle 48^\circ = 250 \angle 78^\circ$

Division = $10 \angle 30^\circ \div 25 \angle 48^\circ = 0.4 \angle -18^\circ$

Currents and voltages in simple circuit elements

(1) Resistor

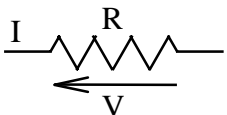


for a sinusoid, consider $i(t) = \text{Real part of } [I_m e^{j(\omega t + \theta)}]$ or $I_m \cos(\omega t + \theta)$

$$\therefore v(t) = \text{Real} [R \cdot I_m e^{j(\omega t + \theta)}] = \text{Real} [V_m \cdot e^{j(\omega t + \theta)}]$$

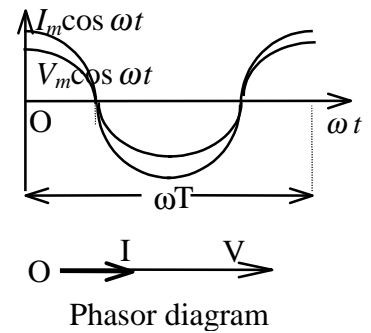
$$v(t) = R \cdot i(t) \quad \text{or} \quad v(t) = R \cdot I_m \cos(\omega t + \theta) = V_m \cos(\omega t + \theta)$$

$$\therefore V_m = R \cdot I_m \quad \text{and} \quad V_m / \sqrt{2} = R \cdot I_m / \sqrt{2}$$



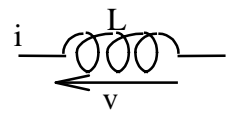
$$\text{i.e. } V = R \cdot I$$

Note: **V** and **I** are rms values of the voltage and current and no additional phase angle change has occurred in the resistor.



Note also that the power dissipated in the resistor **R** is equal to $R \cdot I^2 = V \cdot I$

(2) Inductor

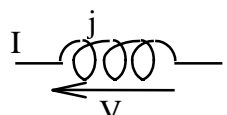


for a sinusoid, consider $i(t) = \text{Real part of } [I_m e^{j(\omega t + \theta)}]$ or $I_m \cos(\omega t + \theta)$

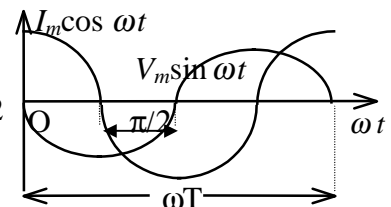
$$\therefore v(t) = \text{Real} [L \cdot \frac{d}{dt} I_m e^{j(\omega t + \theta)}] = \text{Real} [L \cdot j\omega \cdot I_m e^{j(\omega t + \theta)}] = \text{Real} [j V_m e^{j(\omega t + \theta)}]$$

$$v(t) = L \frac{d i(t)}{dt} \quad \text{or} \quad v(t) = L \cdot \frac{d}{dt} I_m \cos(\omega t + \theta) = -L \cdot \omega \cdot I_m \sin(\omega t + \theta) = L \cdot \omega \cdot I_m \cos(\omega t + \theta + \pi/2)$$

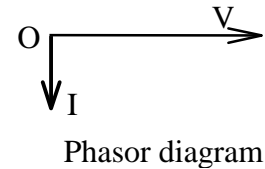
$$= V_m \cos(\omega t + \theta + \pi/2)$$



$$\therefore V_m = \omega L \cdot I_m \quad \text{and} \quad V_m / \sqrt{2} = \omega L \cdot I_m / \sqrt{2}$$



It can be seen that the *rms* magnitude of voltage is related to the *rms* magnitude of current by the multiplying factor ωL . It also seen that the voltage waveform leads the current waveform by 90° or $\pi/2$ radians or that the current waveform lags the voltage waveform by 90° for an inductor .



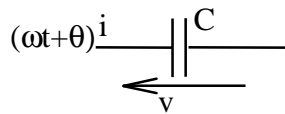
Thus it is usual to write the relationship as $V = j\omega L \cdot I$ or $V = \omega L \cdot I \ 90^\circ$

The impedance Z of the inductance may thus be defined as $j\omega L$, and $V = Z \cdot I$ corresponds to the generalised form of Ohm's Law.

Remember also that the power dissipation in a pure inductor is zero, as energy is only stored and as there is no resistive part in it, but that the product $V \cdot I$ is not zero.

(3) Capacitor

for a sinusoid, consider $i(t) = \text{Real part of } [I_m e^{j(\omega t + \theta)}]$ or $I_m \cos$

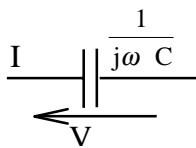


$$v(t) = \text{Real} \left[\frac{1}{C} \int I_m e^{j(\omega t + \theta)} . dt \right] = \text{Real} \left[\frac{1}{C \cdot j\omega} \cdot I_m e^{j(\omega t + \theta)} \right] = \text{Real} \left[\frac{1}{j} \frac{V_m}{\omega} e^{j(\omega t + \theta)} \right]$$

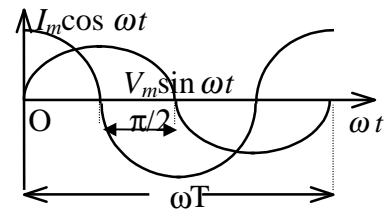
$$v(t) = \frac{1}{C} \int i(t) . dt$$

$$\text{or } v(t) = \frac{1}{C} \int I_m \cos(\omega t + \theta) . dt = \frac{1}{C\omega} \cdot I_m \sin(\omega t + \theta) = \frac{1}{C\omega} \cdot I_m \cos(\omega t + \theta - \pi/2)$$

$$= V_m \cos(\omega t + \theta - \pi/2)$$

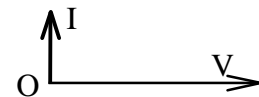


$$\therefore V_m = \frac{1}{\omega C} \cdot I_m \text{ and } V_m/\sqrt{2} = \frac{1}{\omega C} \cdot I_m/\sqrt{2}$$



It can be seen that the *rms* magnitude of voltage is related to the *rms* magnitude of current by the multiplying factor $\frac{1}{\omega C}$.

It also seen that the voltage waveform lags the current waveform by 90° or $\pi/2$ radians or that the current waveform leads the voltage waveform by 90° for a capacitor.



Phasor diagram

Thus it is usual to write the relationship as $V = \frac{1}{j\omega C} \cdot I$ or $V = \frac{1}{\omega C} \cdot I \ 90^\circ$

The impedance Z of the inductance may thus be defined as $\frac{1}{j\omega C}$, and $V = Z \cdot I$ corresponds to the generalised form of Ohm's Law.

Remember also that the power dissipation in a pure capacitor is zero, as energy is only stored and as there is no resistive part in it, but that the product $V \cdot I$ is not zero.

Impedance and Admittance in an a.c. circuit

The *impedance* Z of an a.c. circuit is a complex quantity. It defines the relation between the complex rms voltage and the complex rms current. *Admittance* Y is the inverse of the impedance Z .

$$V = Z \cdot I, \quad I = Y \cdot V \quad \text{where } Z = R + jX, \quad \text{and } Y = G + jB$$

It is usual to express Z in cartesian form in terms of R and X , and Y in terms of G and B .

The real part of the impedance Z is resistive and is usually denoted by a resistance R , while the imaginary part of the impedance Z is called a *reactance* and is usually denoted by a reactance X .

It can be seen that the pure inductor and the pure capacitor has a reactance only and not a resistive part, while a pure resistor has only resistance and not a reactive part.

Thus $Z = R + j0$ for a resistor, $Z = 0 + j\omega L$ for an inductor, and $Z = \frac{1}{j\omega C} = 0 - j\frac{1}{\omega C}$ for a capacitor.

The real part of the admittance Y is a *conductance* and is usually denoted by G , while the imaginary part of the admittance Y is called a *susceptance* and is denoted by B .

Relationships exist between the components of Z and the components of Y as follows.

$$G + jB = Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} \quad \text{so that} \quad G = \frac{R}{R^2 + X^2}, \quad \text{and} \quad B = \frac{-X}{R^2 + X^2}$$

The reverse process can also be similarly done if necessary.

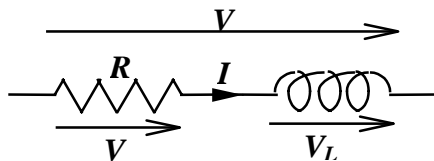
However, it must be remembered that in a complex circuit, G does not correspond to the inverse of the resistance R but its effective value is influenced by X as well as seen above.

Simple Series Circuits

In the case of single elements R , L and C we found that the angle difference between the voltage and the current was either zero, or $\pm 90^\circ$. This situation changes when there are more than one component in a circuit.

R-L series circuit

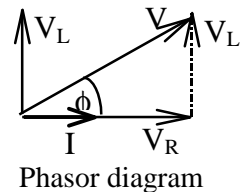
In the series R-L circuit, considering current I as reference



$$V_R = R.I, \quad V_L = j\omega L.I, \quad \text{and} \quad V = V_R + V_L$$

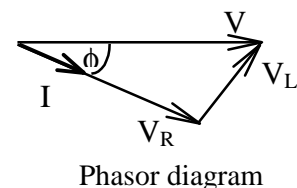
$$\therefore V = (R + j\omega L).I \quad \text{so that}$$

the total series impedance is $Z = R + j\omega L$



The above phasor diagram has been drawn with I as reference. [i.e. I is drawn along the x-axis direction]. The current was selected as reference in this example, because it is common to both the resistance and the inductance and makes the drawing of the circuit diagram easier. In this diagram, the voltage across the resistor V_R is in phase with the current, where as the voltage across the inductor V_L is 90° leading the current. The total voltage V is then obtained by the phasor addition (similar to vector addition) of V_R and V_L .

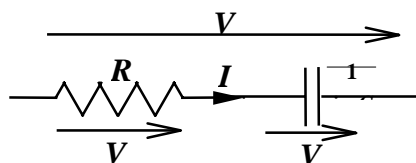
If the total voltage was taken as the reference, the diagram would just rotate as shown. In this diagram, the current is seen to be lagging the voltage by the same angle that in the earlier diagram the voltage was seen to be leading the current. V_L has been drawn from the end of V_R rather than from the origin for ease of obtaining the resultant V from the triangular law.



In an R-L circuit, the current lags the voltage by an angle less than 90° and the circuit is said to be inductive.

Note that the power dissipation can only occur in the resistance in the circuit and is equal to $R \cdot I^2$ and that this is not equal to product $V \cdot I$ for the circuit.

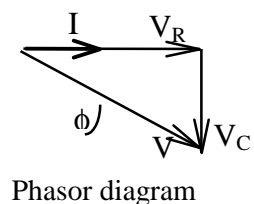
R-C series circuit



In the series R-C circuit,

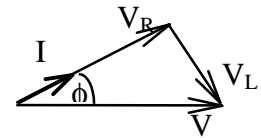
$$V_R = R.I, \quad V_C = \frac{1}{j\omega C} \cdot I = -j\frac{1}{\omega C} \cdot I$$

$$\text{and} \quad V = V_R + V_C$$



$$\therefore V = (R + \frac{1}{j\omega C}).I \quad \text{so that the total series impedance is } Z = R + \frac{1}{j\omega C}$$

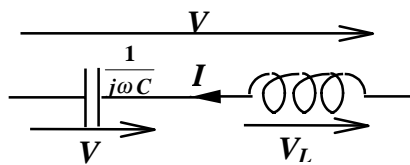
The phasor diagrams has been drawn first with current I as reference and then with voltage V as reference.



In an R-C circuit, the current leads the voltage by an angle less than 90° and the circuit is said to be capacitive.

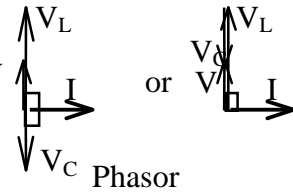
Note that the power dissipation can only occur in the resistance in the circuit and is equal to $R \cdot I^2$ and that this is not equal to product $V \cdot I$ for the circuit.

L-C series circuit



In the series L-C circuit,

$$V_L = j\omega L.I, \quad V_C = \frac{1}{j\omega C}.I = -j \frac{1}{\omega C} V$$



and $V = V_L + V_C$

$$\therefore V = (j\omega L + \frac{1}{j\omega C}).I \quad \text{so that the total series impedance is } Z = j\omega L + \frac{1}{j\omega C} = j\omega L - j \frac{1}{\omega C}$$

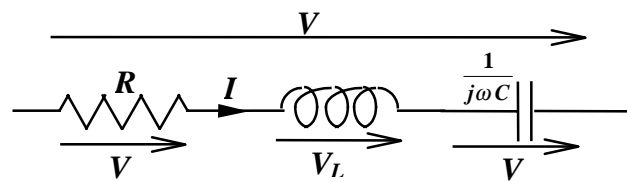
It is seen that the total impedance is purely reactive, and that all the voltages in the circuit are inphase but perpendicular to the current. The resultant voltage corresponds to the algebraic difference of the two voltages V_L and V_C and the direction could be either up or down depending on which voltage is more.

When $\omega L = \frac{1}{\omega C}$ the total impedance of the circuit becomes zero, so that the circuit current for

a given supply voltage would become very large (only limited by the internal impedance of the source). This condition is known as series resonance. In an L-C circuit, the current either lags or leads the voltage by an angle equal to 90° and the resultant circuit is either purely inductive or capacitive.

Note that no power dissipation can occur in the circuit and but that the product $V \cdot I$ for the circuit is non zero.

R-L-C series circuit



In the series R-L-C circuit,

$$V_R = R.I, \quad V_L = j\omega L.I, \quad V_C = \frac{1}{j\omega C}.I = -j \frac{1}{\omega C}.I$$

and $V = V_R + V_L + V_C$

$$\therefore V = (R + j\omega L + \frac{1}{j\omega C}).I \quad \text{so that the total series impedance is } Z = R + j\omega L + \frac{1}{j\omega C}$$

$$Z = R + j(\omega L - \frac{1}{\omega C})$$

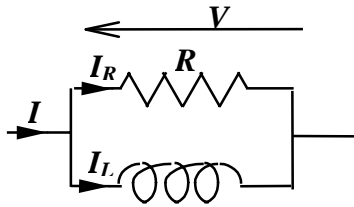
$|Z| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$ has a minimum value at $\omega L = \frac{1}{\omega C}$. This is the series resonance condition.

In an R-L-C circuit, the current can either lag or lead the voltage, and the phase angle difference between the current and the voltage can vary between -90° and 90° and the resultant circuit is either inductive or capacitive.

Note that the power dissipation can only occur in the resistance in the circuit and is equal to $R \cdot I^2$ and that this is not equal to product $V \cdot I$ for the circuit.

Simple Parallel Circuits

R-L parallel Circuit

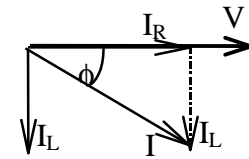


$$\therefore \text{total shunt admittance} = \frac{1}{R} + \frac{1}{j\omega L}$$

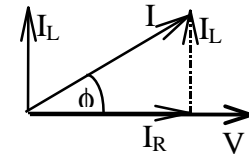
In the parallel R-L circuit, considering V as reference

$$V = R \cdot I_R, \quad V = j\omega L \cdot I_L, \quad \text{and} \quad I = I_R + I_L$$

$$\therefore I = \frac{V}{R} + \frac{V}{j\omega L}$$

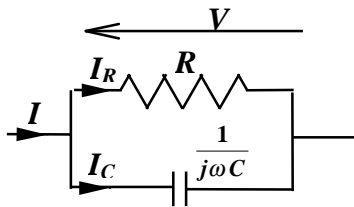


Phasor



Phasor

R-C parallel Circuit



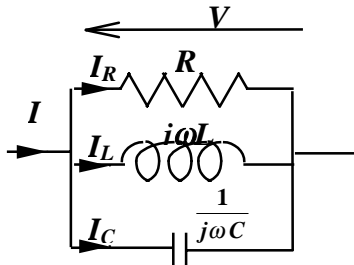
$$\therefore \text{total shunt admittance} = \frac{1}{R} + j\omega C$$

In the parallel R-C circuit, considering V as reference

$$V = R \cdot I_R, \quad I_C = j\omega C \cdot V, \quad \text{and} \quad I = I_R + I_L$$

$$\therefore I = \frac{V}{R} + V \cdot j\omega C$$

R-L-C parallel Circuit



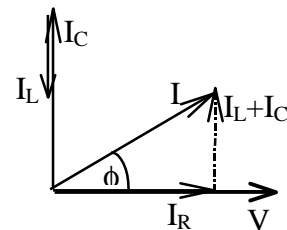
$$\therefore \text{total shunt admittance} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C$$

In the parallel R-L-C circuit, considering V as reference

$$V = R \cdot I_R, \quad V = j\omega L \cdot I_L, \quad I_C = j\omega C \cdot V$$

and $I = I_R + I_L + I_C$

$$\therefore I = \frac{V}{R} + \frac{V}{j\omega L} + V \cdot j\omega C$$



Phasor

As in the case of the series circuit, shunt resonance will occur when $\frac{1}{\omega L} = \omega C$ giving a minimum value of shunt admittance.

Note that even in the case of a parallel circuit, power loss can only occur in the resistive elements and that the product $V \cdot I$ is not usually equal to the power loss.

Power and Power Factor

It was noted that in an a.c. circuit, power loss occurs only in resistive parts of the circuit and in general the power loss is not equal to the product $V \cdot I$ and that purely inductive parts and purely capacitive parts of a circuit did not have any power loss. To account for this apparent discrepancy, we define the product $V \cdot I$ as the **apparent power** S of the circuit.

Apparent power has the unit volt-ampere (VA) and not the watt (W), and watt (W) is used only for the **active power** P of the circuit (which we earlier called the **power**)

$$\text{apparent power } S = V \cdot I$$

Since a difference exists between the **apparent power** and the **active power** we define a new term called the **reactive power Q** for the reactance X .

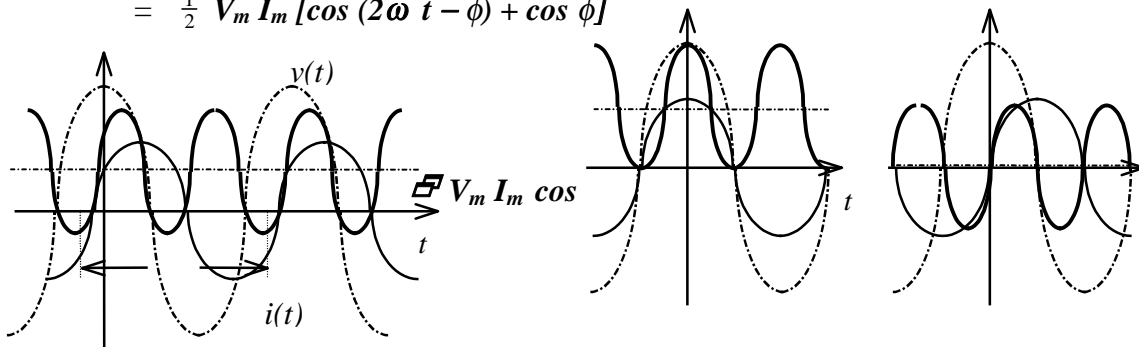
The instantaneous value of power $p(t)$ is the product of the instantaneous value of voltage $v(t)$ and the instantaneous value of current $i(t)$.

i.e. $p(t) = v(t) \cdot i(t)$

If $v(t) = V_m \cos \omega t$ and $i(t) = I_m \cos (\omega t - \phi)$,

where the voltage has been taken as reference and the current lags the voltage by a phase angle ϕ

then $p(t) = V_m \cos \omega t \cdot I_m \cos (\omega t - \phi) = V_m I_m \cdot \frac{1}{2} \cdot 2 \cos \omega t \cdot \cos (\omega t - \phi)$
 $= \frac{1}{2} V_m I_m [\cos (2\omega t - \phi) + \cos \phi]$



current lagging voltage by angle ϕ

inphase

It can be seen that the waveform of power $p(t)$ has a sinusoidally varying component and a constant component.

Thus the average value of power (active power) P would be given by the constant value $\frac{1}{2} V_m I_m \cos \phi$.

$$\text{active power } P = \frac{1}{2} V_m I_m \cos \phi = \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cdot \cos \phi = V \cdot I \cos \phi$$

The term $\cos \phi$ is defined as the **power factor**, and is the **ratio** of the **active power** to the **apparent power**.

Note that for a **resistor**, $\phi = 0^\circ$ so that $P = V \cdot I$

and that for an **inductor**, $\phi = 90^\circ$ lagging (i.e. current is lagging the voltage by 90°) so that $P = 0$

and that for an **capacitor**, $\phi = 90^\circ$ leading (i.e. current is leading the voltage by 90°) so that $P = 0$

For combinations of **resistor**, **inductor** and **capacitor**, P lies between $V \cdot I$ and 0

For an inductor or capacitor, $V \cdot I$ exists although $P = 0$. For these elements the product $V \cdot I$ is defined as the **reactive power Q** . This occurs when the **voltage** and the **current** are **quadrature** (90° out of phase). Thus reactive power is defined as the product of voltage and current components which are quadrature.

This gives

$$\text{reactive power } Q = V \cdot I \sin \phi$$

Unlike in the case of inphase, where the same direction means positive, when quantities are in quadrature there is no natural positive direction. It is usual to define inductive reactive power when the current is lagging the voltage and capacitive reactive power when the current is leading the voltage. It is worth noting that inductive reactive power and capacitive reactive power have opposite signs.

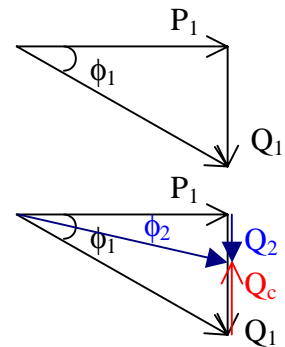
Power factor correction

Although reactive power does not consume any energy, it reduces the power factor below unity. When the power factor is below unity, for the same power transfer P the current required becomes larger and the power losses in the circuit becomes still larger (power loss $\propto I^2$). This is why supply authorities encourage the industries to improve their power factors to be close to unity.

Consider a load with a lagging power factor $\cos \phi_1$.

This will consume an active power P_1 and a reactive power Q_1 as shown in the figure.

If the power factor is to be improved to a new value, $\cos \phi_2$, greater than $\cos \phi_1$, then a certain amount of leading reactive power Q_c must be added. This is usually done by the use of capacitors. If pure capacitors are added, the active power remains unchanged at P_1 , assuming the supply voltage is not affected by the change.



Thus the new reactive power Q_2 would be $Q_1 - Q_c$.

If what is known is P_1 , $\cos \phi_1$ and $\cos \phi_2$, then we have

$$Q_1 = P_1 \tan \phi_1 \text{ and } Q_2 = P_1 \tan \phi_2 \text{ so that } Q_c = Q_1 - Q_2 = P_1 \tan \phi_1 - P_1 \tan \phi_2$$

The reactive power supplied by a capacitor is dependant on its capacitance C and the voltage across it V. Thus

$$Q_c = P_1 \tan \phi_1 - P_1 \tan \phi_2 = V^2 \cdot Y = V^2 \cdot C \omega$$

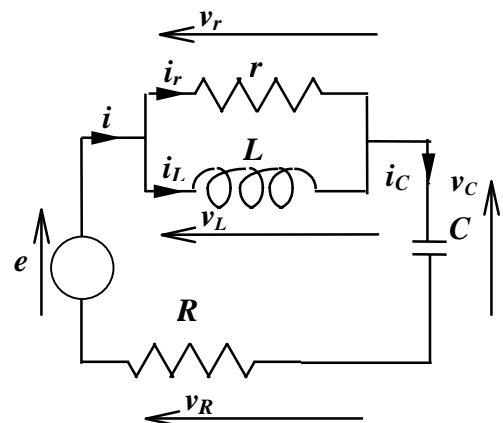
There the value of C can be determined.

Impedance, Admittance and Transfer functions

The behaviour of a bilateral linear circuit is governed by a differential equation, as each of the simple elements are governed either by a constant, differentiation or integration to get the instantaneous voltage $v(t)$ from the instantaneous current $i(t)$.

For example, consider the circuit shown in the figure.

Let us see what the relationship is between the source voltage $e(t)$ and the current $i_r(t)$.



The following equations can be written with $p = \frac{d}{dt}$ using Kirchoff's voltage law, Kirchoff's current Law and Ohm's Law.

$$e(t) = v_r(t) + v_c(t) + v_L(t) - v_R(t)$$

$$v_r(t) = v_L(t), \quad i(t) = i_r(t) + i_L(t) = i_c(t)$$

$$v_r(t) = r \cdot i_r(t), \quad v_L(t) = L p \cdot i_L(t), \quad C p \cdot v_c(t) = i_c(t)$$

Since we are interested in the relationship between $e(t)$ and $i_r(t)$, we eliminate the other variables to give

$$L p \cdot i_L(t) = r \cdot i_r(t), \quad L p \cdot i(t) = L p \cdot [i_r(t) + i_L(t)] = (L p + r) \cdot i_r(t)$$

$$e(t) = r \cdot i_r(t) + v_C(t) + R \cdot i(t)$$

i.e. $C p \cdot e(t) = C p \cdot r \cdot i_r(t) + i(t) + C p \cdot R \cdot i(t)$

$$L p \cdot C p \cdot e(t) = L p \cdot C p \cdot r \cdot i_r(t) + (1 + C p \cdot R) \cdot i_r(t)$$

i.e. $L.C.p^2 \cdot e(t) = [L.C.p^2.r + 1 + C p \cdot R] \cdot i_r(t)$

which is a differential equation involving the up to the second derivative of both $e(t)$ and $i_r(t)$.

or $f_1(p) \cdot e(t) = f_2(p) \cdot i_r(t)$

or $e(t) = Z_r(p) \cdot i_r(t)$

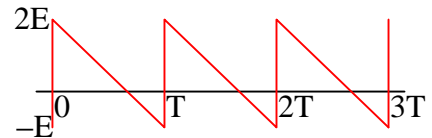
Thus we can see that the source emf $e(t)$ and the current $i_r(t)$ are related by an impedance transfer function of the differential operator p .

In a similar way, the relationship between any current and any voltage would be related by an admittance transfer function of the differential operator p and between any two voltages and any two currents by a transfer gain of the differential operator p .

It is to be noted, that in the case of sinusoidal a.c., the differential operator p may be replaced using the relationship $p = j\omega$.

Example 3

Determine the mean value, average value, peak value, rms value, form factor and peak factor for the waveform shown.



Solution

$$\text{Mean value} = \frac{1}{T} \int_0^T f(t) \cdot dt = \frac{1}{T} \int_0^T E(2 - \frac{3t}{T}) \cdot dt = \frac{E}{T} (2t - \frac{3t^2}{2T}) \Big|_{t=0}^T = \frac{E}{T} (2T - \frac{3T^2}{2T}) = \frac{E}{2}$$

[This result could also have been written by inspection considering areas].

$$\text{Average value} = \frac{1}{T} [\frac{1}{2} \times 2E \times \frac{2}{3}T - (\frac{1}{2} \times (-E) \times \frac{1}{3}T)] = \frac{E}{6T} [5T] = \frac{5E}{6}$$

$$\text{Peak value} = 2E$$

$$\begin{aligned} \text{rms value} &= \sqrt{\frac{1}{T} \int_0^T f^2(t) \cdot dt} = \sqrt{\frac{1}{T} \int_0^T E^2 (2 - \frac{3t}{T})^2 \cdot dt} = \sqrt{\frac{E^2}{T} (2 - \frac{3t}{T})^3 \times \frac{1}{3} (-\frac{T}{3})} \Big|_{t=0}^T \\ &= \sqrt{(-) \frac{E^2}{9} [-1 - 8]} = E \end{aligned}$$

$$\text{form factor} = \frac{E}{\frac{5E}{6}} = \frac{6}{5} = 1.2$$

$$\text{peak factor} = \frac{2E}{\frac{5E}{6}} = \frac{12}{5} = 2.4$$

Example 4

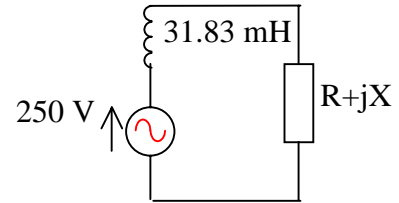
A certain 50 Hz, alternating voltage source has an internal emf of 250 V and an internal inductance of 31.83 mH. If the terminal voltage is to be maintained at 230 V, determine the value of the maximum power that can be delivered to a load ($R + jX$) and the values of R and X under these conditions. Draw also the phasor diagram showing the voltages and currents in the circuit under these conditions.

Solution

at 50 Hz, $x_s = j\omega L = j 2\pi \times 50 \times 31.83 \times 10^{-3} = j 10.00 \Omega$

Across load

Current $I = \frac{250}{j10 + R + jX}$, $|V| = 230 \text{ V}$



active power $P = |I|^2 R = \frac{250^2}{[R^2 + (X + 10)^2]} \cdot R$, voltage $V = (R + jX) \cdot I$,

If we differentiate and obtain the condition for maximum power,
 $\frac{\partial P}{\partial R} = 0$, and $\frac{\partial P}{\partial X} = 0$, we can show that these give the conditions
 $[R^2 + (X + 10)^2] \cdot 1 - R \cdot 2R = 0$ and $(X + 10) = 0$ or $X = -10 \Omega$ and $R = 0$
 Then $P = \infty$, $V = \infty$

$$|V|^2 = 230^2 = (R^2 + X^2) \cdot |I|^2 = (R^2 + X^2) \cdot \frac{250^2}{[R^2 + (X + 10)^2]}$$

$\therefore R^2 + (X + 10)^2 = 1.185 R^2 + 1.1815 X^2$

i.e. $20 X + 10^2 = 0.1815 (R^2 + X^2)$ or $R^2 + X^2 = 5.5104 (10^2 + 20X)$

We can differentiate for maximum power keeping the condition in mind.

i.e. $P = \frac{250^2}{6.5104[10^2 + 20X]} \cdot R$

$\frac{dP}{dR} = 0$, gives the condition $(20X + 100) \cdot 1 - R \cdot 20 \cdot \frac{dX}{dR} = 0$ or $X + 5 = R \frac{dX}{dR}$

also $20 \frac{dX}{dR} + 0 = 0.1815(2R + 2X \frac{dX}{dR})$

so that $20(X + 5) = 0.363R^2 + 0.363X(X + 5)$

i.e. $20X + 100 = 0.363R^2 + 0.363X^2 + 1.815X$

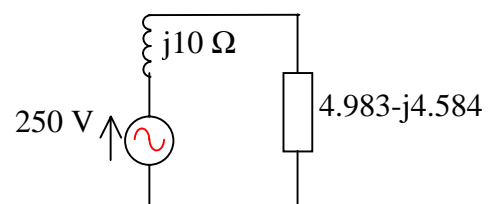
$0.363(R^2 + X^2) = 18.185X + 100$

but $20 X + 100 = 0.1815 (R^2 + X^2) = (9.0925X + 50)$

i.e. $10.9075X = -50$

$X = -4.584 \Omega$, giving $R = 4.983 \Omega$

substituting gives Maximum Power = 5750 W,
 under the given conditions,



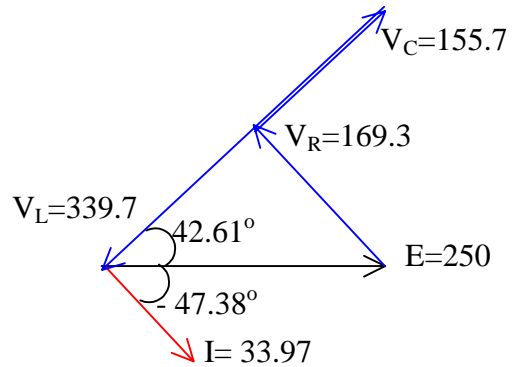
$$I = \frac{250}{j10 + 4.983 - j4.584} = \frac{250}{4.983 + j5.416}$$

i.e. $I = \frac{250}{7.360 \angle 47.38^\circ} = 33.97 \angle -47.38^\circ \text{ A}$

terminal voltage at source (load voltage) = $33.97 \angle -47.38^\circ \times 6.771 \angle -42.61^\circ = 230.0 \angle -90^\circ \text{ V}$

voltage drop across the resistive part of load = $4.983 \times 33.97 \angle -47.38^\circ = 169.3 \angle -47.38^\circ \text{ V}$

voltage drop across the capacitive part of load = $4.584 \angle -90^\circ \times 33.97 \angle -47.38^\circ = 155.7 \angle -137.38^\circ$



Example 5

If in example 4, the load was purely resistive, what would be the value of that resistance for maximum power transfer at 230 V, and what would be the amount of capacitance that must be connected in parallel with the load to achieve this condition.

Solution

The same solution as above can be used, except that we need to find the parallel equivalent of the load.

If G and B are the parallel admittance components,

$$G + jB = \frac{1}{4.983 - j4.584} = \frac{1}{7.360 \angle -47.38^\circ} = 0.092 + j0.1000$$

Therefore the effective value of the resistance = $1/0.0920 = 10.87 \Omega$

Parallel value of capacitance = $B/\omega = 0.1000/(2\pi \times 50) = 318.3 \mu\text{F}$